INSERTION OF A CONTINUOUS FUNCTION AND $X \times I$

by

Ernest P. Lane
If $g$ and $f$ are real-valued functions defined on a topological space $X$ such that $g < f$ (i.e., $g(x) < f(x)$ for each $x$ in $X$), consider the problem of finding necessary and sufficient conditions in order for there to be a continuous function $h$ defined on $X$ such that $g < h < f$. Such conditions seem to have characteristics that keep them from being applicable in some cases. For example, the condition given in Theorem 3.11 of [2] requires that $-g$ must possess the same property as $f$ and that the classes to which $g$ and $f$ belong are closed under sum, supremum, and infimum. The condition given in Theorem 2.1 of [5] places a restriction on $f - g$. This paper gives a necessary and sufficient condition that avoids these problems, but it is in terms of subsets of $X \times I$.

All functions considered are assumed to map into the interval $(0,1)$; $I$ denotes the interval $[0,1]$. (In most situations there is no loss in generality in replacing $\mathbb{R}$ with $(0,1)$.) It is therefore convenient to let $C(X)$ represent the set of all continuous functions from $X$ into $(0,1)$. The characterization that is given below is in terms of the following subsets of $X \times I$: If $k$ maps $X$ into $(0,1)$, let

\[ \text{INSERTION OF A CONTINUOUS FUNCTION AND } X \times I \]

Ernest P. Lane

\[ \text{1This work was supported by a released teaching load research grant from Appalachian State University.} \]
\[
L_k = \{(x,t) \in X \times I: t \leq k(x)\}
\]
and
\[
U_k = \{(x,t) \in X \times I: t \geq k(x)\}.
\]

It is observed in [6] that if \(k\) is continuous then \(L_k\) and \(U_k\) are zero sets in \(X \times I\); this will be used below in a proof. It is convenient to use the following terminology. If \(k\) is a real-valued function then \(k\) is \(z\)-lower semicontinuous (z-lsc) [respectively, \(z\)-upper semicontinuous (z-usc)] if for each real number \(r\), \(\{x \in X: k(x) \leq r\}\) [respectively, \(\{x \in X: k(x) \geq r\}\)] is a zero set. (These functions have been used in [1] and in [8].)

**Proposition 1.** If \(g\) and \(f\) are functions from \(X\) into \((0,1)\) such that \(g < f\) and such that \(L_g\) and \(U_f\) are completely separated in \(X \times I\), then there is a continuous function \(h\) on \(X\) such that \(g < h < f\).

**Proof.** If \(g\) and \(f\) are functions from \(X\) into \((0,1)\) such that \(g < f\) and such that \(L_g\) and \(U_f\) are completely separated, let \(F\) be a continuous function from \(X \times I\) into \(I\) such that if \((x,t)\) is in \(L_g\) then \(F(x,t) = 0\) and if \((x,t)\) is in \(U_f\) then \(F(x,t) = 1\). Define functions \(f_1\) and \(g_1\) from \(X\) into \((0,1)\) as follows:

\[
f_1(x) = \inf\{t \in I: (\{x\} \times [t,1]) \subseteq F^{-1}(\{1\})\}
\]
and

\[
g_1(x) = \sup\{t \in I: (\{x\} \times [0,t]) \subseteq F^{-1}(\{0\})\}.
\]

Since \(L_g \subseteq F^{-1}(\{0\})\) and \(U_f \subseteq F^{-1}(\{1\})\), \(g \leq g_1\) and \(f_1 \leq f\). Since \(F\) is continuous it is easy to show that \(g_1 < f_1\). Also, \(g_1\) is zusc and \(f_1\) is zlsc; the outline of an argument to show that \(f_1\) is zlsc follows: If \(r\) is any real number
in \((0,1)\), define a function \(F_r\) from \(X\) into \(I\) by
\[
F_r(x) = \inf\{F(x,t) : r \leq t \leq 1\}.
\]
Then \(F_r\) is a continuous function from \(X\) into \(I\). Next observe that
\[
\{x \in X : f_1(x) \leq r\} = \{x \in X : F_r(x) = 1\}.
\]
For if \(f_1(x) \leq r\) then \(\{x\} \times [r,1] \subseteq F^{-1}(\{1\})\) and hence \(F_r(x) = 1\). Conversely, if \(F_r(x) = 1\) then \(F(x,t) = 1\) for all \(t\) such that \(r \leq t \leq 1\) and consequently \(\{x\} \times [r,1] \subseteq F^{-1}(\{1\})\); thus \(f_1(x) \leq r\). Since \(F_r\) is continuous, \(\{x \in X : F_r(x) = 1\}\) is a zero set. Since \(\{x \in X : f_1(x) \leq r\}\) is a zero set for any \(r\) in \((0,1)\), it follows from the definition that \(f_1\) is zlsc. A similar argument shows that \(g_1\) is zusc. Thus \(g \leq g_1 \leq f_1 \leq f\), \(g_1\) is zusc and \(f_1\) is zlsc; by Proposition 6.1 of [1], there is a continuous function \(h\) defined on \(X\) such that \(g_1 < h < f_1\). Hence \(g < h < f\) and this concludes the proof.

Remark. In the above proof the zero set \(F^{-1}(\{1\})\) in \(X \times I\) is used to define a zlsc function on \(X\); in the following it is observed that a zlsc function on \(X\) corresponds to a zero set in \(X \times I\). Tong [9] proves that a space \(X\) is perfectly normal if and only if each lower semicontinuous function on \(X\) is a pointwise limit of an increasing sequence of continuous functions. His proof is inductive and involves the complete separation of certain subsets of \(X\). It is straightforward to observe that this proof yields the following result: If \(f\) is a zlsc function on any topological space \(X\), there is an increasing sequence of continuous functions on \(X\) whose pointwise limit is \(f\). (The converse of
this is immediate from the definition of zlsc.) Consequently, if \( f \) is zlsc then \( U_f \) is a countable intersection of zero sets, and hence \( U_f \) is a zero set in \( X \times I \).

The main result of the paper follows from Proposition 1.

**Theorem 1.** Let \( L(X) \) and \( U(X) \) be classes of functions from \( X \) into \((0,1)\) such that \( C(X) \subset L(X) \) and \( C(X) \subset U(X) \).

The following are equivalent:

(i) For any \( g \in U(X) \) and any \( f \in L(X) \) such that \( g < f \) there is an \( h \in C(X) \) such that \( g < h < f \).

(ii) For any \( g \in U(X) \) and any \( f \in L(X) \) such that \( g < f \) then \( L_g \) and \( U_f \) are completely separated in \( X \times I \).

**Proof.** That (ii) implies (i) follows from Proposition 1 (and does not require the hypothesis that \( C(X) \subset L(X) \) and \( C(X) \subset U(X) \)). Conversely, suppose that (i) is satisfied.

If \( g \in U(X) \), \( f \in L(X) \) and \( g < f \), then by (i) there exists \( h \in C(X) \) such that \( g < h < f \). Since \( C(X) \subset L(X) \), apply (i) to \( g \) and \( h \); there exists \( k \in C(X) \) such that \( g < k < h \).

Since \( k \) and \( h \) are continuous \( L_k \) and \( U_h \) are zero sets; since \( k < h \) it follows that \( L_k \) and \( U_h \) are disjoint. Since \( L_g \subset L_k \) and \( U_f \subset U_h \), the sets \( L_g \) and \( U_f \) are completely separated.

An application of this theorem is given below that involves normal semicontinuous functions. If \( f_* \) [respectively, \( f^* \)] denotes the lower [respectively, upper] limit function of \( f \) then \( f \) is normal lower semicontinuous (nlsc) in case \( f = (f^*)_* \) and \( f \) is normal upper semicontinuous (nusc) in case \( f = (f_*)^* \). Since any continuous function is nlsc and nusc, the following is immediate from Theorem 1.
Corollary 1. Let \( L(X) \) [respectively, \( U(X) \)] denote the \( nls \)c [respectively, \( nusc \)] functions from \( X \) into \((0,1)\).

The following are equivalent:

(i) For any \( g \) in \( U(X) \) and any \( f \) in \( L(X) \) such that \( g < f \) there is an \( h \) in \( C(X) \) such that \( g < h < f \).

(ii) For any \( g \) in \( U(X) \) and any \( f \) in \( L(X) \) such that \( g < f \) then \( L_g \) and \( U_f \) are completely separated in \( X \times I \).

The proof of Theorem 3.2 of [6] shows that if \( X \) is any space such that \( g \ nusc \) and \( f \ nls \)c and \( g < f \) imply that \( L_g \) and \( U_f \) are completely separated in \( X \times I \), then there is a continuous function \( h \) on \( X \) such that \( g < h < f \). The proof of this uses the result that for \( g \ nusc \) and \( f \ nls \)c then \( L_g \) and \( U_f \) are regular closed sets in \( X \times I \). The general converse of Theorem 3.2 of [6] remains open: Does condition (i) of Corollary 1 (for \( g \ nusc \) and \( f \ nls \)c) imply that arbitrary disjoint regular closed subsets of \( X \times I \) are completely separated?

If \( X \times I \) satisfies the equivalent conditions of Theorem 1 for various classes of functions, the corresponding characterization of the space \( X \) is known. If \( L(X) \) (respectively, \( U(X) \)) is the class of lower (respectively, upper) semicontinuous functions on \( X \), the conditions of Theorem 1 are equivalent to \( X \) is normal and countably paracompact. (This follows from Theorem 4 of [3] or from Theorem 2 of [4].) If \( L(X) \) (respectively, \( U(X) \)) is the class of upper (respectively, lower) semicontinuous functions on \( X \), then the conditions of Theorem 1 are equivalent to \( X \) is an extremally disconnected P-space that satisfies Baire's condition. (See Proposition 6.11 of [1].) For an
extensive list of corresponding characterizations of $X$ for various cases of the classes $L(X)$ and $U(X)$, see Theorem 4.2 of [7].

References


Appalachian State University

Boone, NC 28608