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FUNCTIONAL BASES FOR SUBSETS OF C* (X)

B. J. Ball and Shoji Yokura

Let X be a completely regular space and, as usual, let C(X) denote the set of all continuous real valued functions on X and C*(X) the set of all bounded functions in C(X). If $G \subset C^*(X)$, we will call a subset F of G a *functional base* for G if each element of G can be expressed as a continuous function of elements of F; if n is a cardinal number and each element of G is a continuous function of (at most) n elements of F, then F is said to be a functional base of *order* n for G.

Making strong use of results from our previous papers [1,2], we first show that if G determines a compactification of X (in the sense of [1]), then every functional base for G determines the same compactification; the converse is false. If, however, G *generates* a compactification of X, then a subset F of G is a functional base for G if and only if F generates the same compactification of X.

We next assume that X is compact and consider functional bases for the set C(X) of *all* real functions on X. For each compact Hausdorff space X, let $m(X)$ (resp., $m_n(X)$) denote the minimum cardinality of the functional bases (of order n) for C(X). We show that $m(X)$, if infinite, is equal to the weight of X, and that the inequalities $m_n(X) \geq m_k(X) \geq m(X)$ hold whenever $n \leq k$. This latter condition implies that there are at most a finite number of distinct values for $m_n(X)$, n ranging over all cardinal numbers. Assuming the generalized continuum hypothesis, there are at most two such values, namely $m(X)$ and $|C(X)|$. The continuum hypothesis gives the same result for $m_n(X)$, $n = 1,2,\cdots$; indeed, the continuum hypothesis is equivalent to the assertion that $m_1(X) = |C(X)|$ if X is not embeddable in $\mathbf{R}$.

1. Definitions and Preliminary Results

Most of our usage is standard or agrees with that of [1,2]. We repeat here the most often used definitions and conventions, along with a few basic results.

Unless the contrary is specifically stated, all spaces considered will be assumed to be completely regular.

Lower case italic letters always denote cardinal numbers; if A is a set, $|A|$ denotes the cardinal number of A.

For each cardinal number n, represent $\mathbf{R}^n$ as $\prod_{\nu\in N} \mathbf{R}_\nu$, where $|N| = n$ and $\mathbf{R}_\nu = \mathbf{R}$ for each ν, and let $\pi_\nu\colon \mathbf{R}^n \to \mathbf{R}$ denote the ν^{th} projection map. If g is a function from X into $\mathbf{R}^n$, then $\pi_\nu \circ g\colon X \to \mathbf{R}$ is called the ν^{th} *coordinate function* of g and will be denoted g_ν. It is well known that $g\colon X \to \mathbf{R}^n$ is continuous if and only if $g_\nu\colon X \to \mathbf{R}$ is continuous for each $\nu \in N$. Moreover, if X is a subspace of $\hat{X}$, then a map $\hat{g}\colon \hat{X} \to \mathbf{R}^n$ is an extension of a map $g\colon X \to \mathbf{R}^n$ if and only if for each $\nu \in N$, $\hat{g}_\nu\colon \hat{X} \to \mathbf{R}$ is an extension of $g_\nu\colon X \to \mathbf{R}$. This fact will be used frequently, often without mention.

If $F = \{f_\nu | \nu \in N\}$ is an indexed subset of $C^*(X)$, then the *evaluation map* of F *corresponding to the given indexing* is the map $e_F\colon X \to \mathbf{R}^n$, $n = |N|$, defined by the condition

$\pi_\nu \circ e_F = f_\nu$ for each $\nu \in N$. Note that this definition is more general than the usual one. In particular, e_F is not uniquely determined by F, nor is its range, since the indexing need not be 1-1. (Our "evaluation map" is the same as the *diagonal map* $\Delta_{\nu\in N} f_\nu$ defined by Engelking [4, p. 110].)

A compactification αX of X is said to be *generated* by a set $F \subset C^*(X)$ if (some) e_F is an embedding and αX is equivalent to $\overline{e_F(X)}$; X is *determined* by F if αX is the smallest compactification of X to which each element of F extends. If αX is either generated or determined by F, then αX will be denoted $\alpha_F X$.

If X and Y are topological spaces, then C(X,Y) denotes the set of all maps (continuous functions) from X into Y; $C(X,\mathbf{R})$ is denoted C(X), and C*(X) denotes the set of all bounded functions in C(X). If $F \subset C(X,Y)$ and $G \subset C(Y,Z)$, then $G \circ F = \{g \circ f | f \in F, g \in G\}$.

For each compactification αX of X, C_α is the set of all elements of C*(X) which have extensions to αX; if $f \in C_\alpha$, the extension of f to αX is denoted by f^α and if $F \subset C_\alpha$, then $F^\alpha = \{f^\alpha | f \in F\}$.

1.1 *Definition*. If $F \subset C^*(X)$ and n is a cardinal number, then $F_n = \{g: X \to \mathbf{R}^n | g_\nu \in F$ for each $\nu \in N\}$ and $M^n(F) = C(\mathbf{R}^n,\mathbf{R}) \circ F_n = \{\phi \circ g | g \in F_n, \phi: \mathbf{R}^n \to \mathbf{R}\}$.

As in [2], we let $e_F: X \to \mathbf{R}^m$ be an arbitrarily chosen evaluation map for F and let $M_F = \{\phi \circ e_F | \phi: \mathbf{R}^m \to \mathbf{R}\}$. (It is shown in [2] that M_F is independent of the choice of e_F.) We can now give a more precise definition of functional bases.

1.2 *Definition.* A *functional base* (of order n) for a subset G of C*(X) is a set F such that $F \subset G \subset M_F$ (resp., $F \subset G \subset M^n(F)$).

1.3 *Lemma. For each* $F \subset C^*(X)$ *and each cardinal number* n, $M^n(F) = \cup\{M_G | G \subset F, |G| \leq n\}$.

Proof. If $p = \phi \circ g \in M^n(F)$, where $g \in F_n$ and $\phi: \mathbf{R}^n \to \mathbf{R}$, then g is an evaluation map for the set $G = \{g_\nu | \nu \in N\}$, so $p \in M_G$. Since $G \subset F$ and $|G| \leq n$, it follows that $M^n(F) \subset \cup\{M_G | G \subset F, |G| \leq n\}$.

Conversely, suppose $p \in M_G$, where $G \subset F$ and $|G| \leq n$. Since $|\bar{G}| \leq n$, there is an evaluation map e_G for G with range $\mathbf{R}^n$. Since $p \in M_G$, $p = \phi \circ e_G$ for some $\phi: \mathbf{R}^n \to \mathbf{R}$. Since $G \subset F$, $e_G \in F_n$ and hence $p \in M^n(F)$. It follows that $\cup\{M_G | G \subset F, |G| \leq n\} \subset M^n(F)$.

The following result is an immediate consequence of Lemma 1.3.

1.4 *Corollary. For any* $F \subset C^*(X)$,

(i) $n \leq k$ *implies* $M^n(F) \subset M^k(F)$

(ii) $M^n(F) \subset M_F$ *for all* n

(iii) *if* $n \geq |F|$, *then* $M^n(F) = M_F$.

As in [1,2], we let $\mathcal{S}(X)$ denote the collection of all subsets of C*(X) which separate points and closed sets, $\mathcal{E}(X)$ the collection of all those which generate compactifications of X and $\mathcal{D}(X)$ those which determine compactifications of X. We always have $\mathcal{S}(X) \subset \mathcal{E}(X) \subset \mathcal{D}(X)$ and, in general, both inclusions are proper.

2. Functional Bases for Elements of $\mathcal{D}(X)$ and $\mathcal{E}(X)$

Not much can be said about functional bases for arbitrary subsets of $C^*(X)$, but results from [1,2] lead to some properties of functional bases for sets which determine or generate compactifications.

2.1 *Lemma. If* αX *is a compactification of* X *and* $F \subset C_\alpha$, *then for every* n, $M^n(F) \subset C_\alpha$ *and* $(M^n(F))^\alpha = M^n(F^\alpha)$. *In particular,* $M_F \subset C_\alpha$ *and* $M_F = M_{F^\alpha}$.

Proof. Suppose $p = \phi \circ g \in M^n(F)$, where $g \in F^n$ and $\phi : \mathbf{R}^n \to \mathbf{R}$. If $g^\alpha : \alpha X \to \mathbf{R}$ is defined by setting $(g^\alpha)_\nu = (g_\nu)^\alpha$ for every $\nu \in N$, then g^α is an extension of g and hence $\phi \circ g^\alpha$ is an extension of $p = \phi \circ g$, so $p \in C_\alpha$. Since g^α, as defined above, belongs to $(F^\alpha)_n$, and for any $h \in (F^\alpha)_n$, $h|X \in F_n$, it easily follows that $(M^n(F))^\alpha = M^n(F^\alpha)$. Since $M_F = M^{|F|}$ by Lemma 1.4, we also have $M_F \subset C_\alpha$ and $M_F^\alpha = M_{F^\alpha}$ for any $F \subset C_\alpha$.

2.2 *Theorem. If* G *is a subset of* $C^*(X)$ *which determines a compactification of* X, *then every functional base for* G *determines the same compactification of* X.

Proof. Suppose G determines the compactification αX of X and $F \subset G \subset M_F$. By [1, Theorem 2.1], $G \subset C_\alpha$ and G^α separates points of $\alpha X - X$. Since $F \subset G$, $F \subset C_\alpha$ and hence by Lemma 2.1, $M_F \subset C_\alpha$. Since $G \subset M_F$, $G^\alpha \subset M_F^\alpha$ and hence M_F^α separates points of $\alpha X - X$ since G^α does so. Applying [1, Theorem 2.1] again, we conclude that M_F determines the compactification αX. By [1, Theorem 3.2], F determines the same compactification that M_F does, so $\alpha_F X = \alpha X$.

Remark. It is not true that if G determines a compactification of X, then every subset of G which determines the same compactification of X is a functional base for G. To see this, let X be locally compact and let $G = C_\omega$, where ωX is the one-point compactification of X. If $F = \{f\}$, where f is a constant map of X into $\mathbf{R}$, then $F \subset G$ and $\alpha_F X = \omega X = \alpha_G X$, but $M_F = \{k\colon X \to \mathbf{R} \mid k \text{ is constant}\}$ and $G \not\subset M_F$. We do have the following result, however.

2.3 *Theorem. If* G *is a subset of* $C^*(X)$ *which generates a compactification* αX *of* X, *then a subset* F *of* G *is a functional base for* G *if and only if* F *generates* αX.

Proof. Suppose first that F is a functional base for G; i.e., $F \subset G \subset M_F$. It follows from [2, Theorem 2.3] that $G \subset C_\alpha$ and G^α separates points of αX. Since $F \subset G$, $F \subset C_\alpha$ and hence by Lemma 2.1 $M_F \subset C_\alpha$. Since $G \subset M_F$ and G^α separates points of αX, M_F^α separates points of αX and since $M_F^\alpha = M_{F^\alpha}$ by Lemma 2.1, M_{F^α} separates points of αX. This readily implies that F^α also separates points of αX. Applying [2, Theorem 2.3] again, we conclude that $F \in \mathcal{E}(X)$ and $\alpha_F X = \alpha X = \alpha_G X$, as required.

Now suppose $F \subset G$, $F \in \mathcal{E}(X)$ and $\alpha_F(X) = \alpha_G X = \alpha X$. By [2, Theorem 2.4], we have $M_F = C_\alpha$. Hence $G \subset C_\alpha = M_F$, so F is a functional base for G.

3. Cardinalities of Functional Bases for C_α

Given a compactification αX of a completely regular space X, the minimum cardinality of the functional bases (of order n) for all of C_α would seem to be of some interest.

Since it follows immediately from Lemma 2.1 that a set $F \subset C_\alpha$ is a functional base (of order n) for C_α if and only if F^α is a functional base (of order n) for $C(\alpha X)$, it is sufficient to restrict attention to the case in which X is compact.

3.1 *Definition.* For each compact Hausdorff space X, $m(X) = \min\{|F|: M_F = C(X)\}$ and for each cardinal number n, $m_n(X) = \min\{|F|: M^n(F) = C(X)\}$.

Additionally, for each $F \subset C(X)$ we let $M^\infty(F) = \bigcup_{n=1}^{\infty} M^n(F)$ and let $m_\infty(X) = \min\{|F|: M^\infty(F) = C(X)\}$.

We first note that if $n \leq k$, then by Lemma 1.4 $M^n(F) \subset M^k(F) \subset M_F$ for every $F \subset C(X)$ and hence $m_n(X) \geq m_k(X) \geq m(X)$. Since for every positive integer n, $M^n(F) \subset M^\infty(F) \subset M^{\aleph_0}(F)$, we also have $m_n(X) \geq m_\infty(X) \geq m_{\aleph_0}(X)$, $n = 1,2,\cdots$. In particular, since there does not exist an infinite decreasing sequence of cardinal numbers, there are only a finite number of distinct values for $m_n(X)$, n ranging over all cardinal numbers. Moreover, if $n_1 = |C(X)|$, then for each $F \subset C(X)$, $|F| \leq n_1$ and hence by Lemma 1.4 $M^{n_1}(F) = M_F$ and therefore $m(X) = m_{n_1}(X)$. Thus $m(X) = \min\{m_n(X)\}$.

3.2 *Theorem. For each compact Hausdorff space* X *and each cardinal number n,* X *is embeddable in $\mathbf{R}^n$ (and hence in I^n) if and only if $m(X) \leq n$, and this is true if and only if $m_n(X) \leq n$.*

Proof. If $m(X) \leq n$, there is a subset F of C(X) such that $|F| \leq n$ and $M_F = C(X)$. Since $|F| \leq n$, there is an evaluation map e_F for F with range $\mathbf{R}^n$. Then $M_F = \{\phi \circ e_F | \phi: \mathbf{R}^n \to \mathbf{R}\} = C(X)$.

If a and b are distinct elements of X, there is a map $h: X \to \mathbf{R}$ with $h(a) \neq h(b)$. Since $h = \phi \circ e_F$ for some $\phi: \mathbf{R}^n \to \mathbf{P}$, it follows that $e_F(a) \neq e_F(b)$. Hence $e_F: X \to \mathbf{R}^n$ is injective and therefore is an embedding since X is compact.

Conversely, suppose $f: X \to \mathbf{R}^n$ is an embedding and let $F = \{f_\nu | \nu \in N\}$. If $h \in C(X)$, the map $h \circ f^{-1}: f(X) \to \mathbf{R}$ can be extended to a map $\phi: \mathbf{R}^n \to \mathbf{R}$, so $h = \phi \circ f$. Since f is an evaluation map for F, it follows that $h \in M_F$. Hence $C(X) \subset M_F$ and since also $M_F \subset C(X)$, we have $M_F = C(X)$. Hence $m(X) \leq |F| \leq n$.

With regard to the second assertion of the lemma, since $m(X) \leq m_n(X)$ for every n, $m_n(X) \leq n$ certainly implies $m(X) \leq n$. For the converse, suppose $m(X) \leq n$ and let F be a subset of C(X) such that $|F| \leq n$ and $M_F = C(X)$. Since $n \geq |F|$, Lemma 1.4 implies that $M^n(F) = M_F$, and hence $m_n(X) \leq |F| \leq n$.

3.3 *Corollary. For every compact Hausdorff space* X, $m(X) = \min\{n | X$ *is embeddable in* $I^n\}$; *in particular, if* $m(X)$ *is infinite, then* $m(X)$ *is the weight of* X.

The preceding theorem and its corollary imply that a compact Hausdorff space X is metrizable if and only if $m_{\aleph_0}(X)$ is countable. We next show that X is metrizable and finite dimensional if and only if $m_\infty(X)$ is countable.

If X and Y are topological spaces, we let $C(X,Y)$ denote, as usual, the set of all continuous functions from X to Y. If X, Y, Z are topological spaces, we will say that a set $F \subset C(X,Z)$ is *generated by* a set $G \subset C(X,Y)$ if every element of F factors through Y with first factor in G; i.e., for each $f: X \to Z$ in F, there is a $g: X \to Y$ in G and an $h: Y \to Z$ in $C(Y,Z)$ such that $f = h \circ g$.

3.4 *Theorem. A compact Hausdorff space* X *is embeddable in a normal space* Y *if and only if* $C(X)$ *is generated by a countable subset* G *of* $C(X,Y)$.

Proof. Suppose $g: X \to Y$ is an embedding. It follows as in the proof of Theorem 3.2 that $C(X) = \{\phi \circ g | \phi: Y \to \mathbf{R}\}$, and hence $C(X)$ is generated by $G = \{g\} \subset C(X,Y)$.

Conversely, suppose $G = \{g_1, g_2, \cdots\}$ is a countable subset of $C(X,Y)$ which generates $C(X,\mathbf{R})$. Suppose no element of G is injective. For each i, let a_i, b_i be distinct points of X such that $g_i(a_i) = g_i(b_i)$, and let $A = \bigcup_{i=1}^{\infty} \{a_i, b_i\}$. Then A is a countable subset of X and no element of G is $1 - 1$ on A. By [5, Theorem 1] or [3, Theorem 5.32], there is a map $f: X \to \mathbf{R}$ which is $1 - 1$ on A and which therefore cannot be of the form $h \circ g_i$ for any $g_i \in G$, contrary to the assumption that G generates $C(X,\mathbf{R})$. Hence some element of G must be injective and since X is compact, it follows that X is embeddable in Y.

3.5 *Theorem. A compact Hausdorff space* X *is metrizable and finite dimensional if and only if* $m_\infty(X)$ *is countable.*

Proof. Suppose $m_\infty(X) \leq \aleph_0$ and let F be a countable subset of $C(X)$ such that $M^{\infty}(F) = C(X)$. Let $Y = \oplus\{Y_n | n = 1,2,\cdots\}$ be the topological sum (disjoint union) of spaces Y_n, with $Y_n \approx \mathbf{R}^n$ for each n. For each n, let $j_n: \mathbf{R}^n \to Y$ be an injection with $j_n(\mathbf{R}^n) = Y_n$, and let $G_n = \{j_n \circ g | g \in F_n\}$. Finally, let $G = \bigcup_{n=1}^{\infty} G_n$. Since F is countable, each F_n is countable and it follows that G is countable.

If $h \in C(X)$, then since $M^{\infty}(F) = C(X)$, $h \in M^n(F)$ for some n and therefore $h = \phi \circ g$ for some $g \in F_n$ and some $\phi: \mathbf{R}^n \to \mathbf{R}$. Since Y_n is a closed subset of the normal space Y, the map $\phi \circ j_n^{-1}: Y_n \to \mathbf{R}$ can be extended to a map $\phi': Y \to \mathbf{R}$. Since $\phi' \circ j_n \circ g = \phi \circ g = h$ and $j_n \circ g \in G$, it follows that G generates $C(X)$. Hence by Theorem 3.4, X is embeddable in Y.

Since X is compact and each Y_i is open in Y, X is embeddable in $\bigcup_{i=1}^{k} Y_i$ for some k, which implies that X is metrizable and finite dimensional.

3.6 *Theorem.* *If* n *is a positive integer and* $m_n(X) \leq \aleph_0$, *then* $m_n(X) \leq n$.

Proof. If F is a countable subset of $C(X)$ such that $M^n(F) = C(X)$, then F_n is a countable set of maps of X into $\mathbf{R}^n$ which generates $C(X)$ and thus, by Theorem 3.4, X is embeddable in $\mathbf{R}^n$. Theorem 3.2 then implies that $m_n(X) \leq n$.

3.7 *Theorem.* *For each positive integer* n, $m_n(X)$ *is either finite or uncountable, and if* $m_n(X)$ *is finite, then* $m_n(X) = m(X)$; *the same is true of* $m_{\infty}(X)$.

Proof. It follows directly from Theorem 3.6 that $m_n(X)$ is either finite or uncountable, and Theorem 3.5

implies that if $m_\infty(X)$ is countable, then $m_k(X)$ is finite for some positive integer k; since $m_k(X) \geq m_\infty(X)$, it follows that $m_\infty(X)$ is finite if it is countable.

Now suppose $m_n(X)$ is finite for some positive integer n. Since $m_n(X) \geq m(X)$, $m(X)$ is also finite, say $m(X) = n_o$. By Theorem 3.2, X is embeddable in $\mathbf{R}^{n_o}$ and hence $m_{n_o}(X) \leq n_o$. Since $n_o \leq n$, $m_{n_o}(X) \geq m_n(X)$. Thus $m(X) = n_o \geq m_{n_o}(X) \geq m_n(X) \geq m(X)$, so $m_n(X) = m(X)$.

If $m_\infty(K)$ is finite, then by Theorem 3.5 $m_n(X)$ is finite for some positive integer n. Applying the preceding argument, we have

$$m(X) = m_n(X) \geq m_\infty(X) \geq m(X), \text{ so } m_\infty(X) = m(X).$$

The preceding theorem shows that for each positive integer n, $m_n(X)$ is either uncountable or equal to $m(X)$, and similarly for $m_\infty(X)$. Assuming the continuum hypothesis (CH), we can be more specific about the possible values of $m_n(X)$ and $m_\infty(X)$.

3.8 *Theorem* (CH). *For each positive integer* n, *either* $m_n(X) = m(X)$ *or* $m_n(X) = |C(X)|$; *the same is true of* $m_\infty(X)$.

Proof. First let $m = m_\infty(X)$; by the preceding result, if $m \leq \aleph_o$ then $m_\infty(X) = m(X)$, so assume $m > \aleph_o$.

For each positive integer n, $M^n(F) = \{\phi \circ g | g \in F_n$ and $\phi \in C(\mathbf{R}^n,\mathbf{R})\}$ and hence $|M^n(F)| \leq |F_n| \cdot |C(\mathbf{R}^n,\mathbf{R})|$. Since $|F_n| = |F|^n = m^n = m$ and $|C(\mathbf{R}^n,\mathbf{R})| = c$, we have $|M^n(F)| \leq m \cdot c$ for $n = 1,2,\cdots$. Since $C(X) = M^\infty(F) = \bigcup_{n=1}^{\infty} M^n(F)$, $|C(X)| \leq \sum_{n=1}^{\infty} m \cdot c = \aleph_o \cdot m \cdot c = m \cdot c$. Since $m > \aleph_o$ by hypothesis, CH implies that $m \geq c$ and hence

$m \cdot c = m$. Thus $|C(X)| \leq m$ and since $F \subset C(X)$ and $|F| = m$, $m \leq |C(X)|$. Hence $m = |C(X)|$.

The argument to show that $m_n(X) > \aleph_0$ implies $m_n(X) = |C(X)|$ is similar but simpler.

Assuming the generalized continuum hypothesis (GCH) gives an analogous result for $m_n(X)$, n an arbitrary cardinal. We will need the following easy lemma.

3.9 *Lemma. If* X *is an infinite compact Hausdorff space of weight* w, *then* $|C(X)| \leq 2^w$ *and hence for every* n, $m_n(X) \leq 2^w$.

Proof. Since X has a basis of cardinality w, there is a dense subset D of X with $|D| \leq w$. Since any continuous function on X is determined by its values on D, it follows that $|C(X)| \leq 2^w$. Since $m_n(X) \leq |C(X)|$, $m_n(X) \leq 2^w$.

3.10 *Theorem* (GCH). *For every cardinal number* n, *either* $m_n(X) = m(X)$ *or* $m_n(X) = |C(X)|$.

Proof. We may assume that n is infinite. If $m(X)$ is finite, then X is embeddable in $\mathbf{R}^n$ for some positive integer n and hence $m_n(X) \leq n$. Applying Theorem 3.8, we have $m(X) = m_n(X) \geq m_n(X) \geq m(X)$, so $m_n(X) = m(X)$. Hence we may also assume that $m(X)$ is infinite, and therefore $m(X) = w$, the weight of X.

From Lemma 3.9 we have $2^w \geq m_n(X)$ and since $m_n(X) \geq m(X) = w$, $2^w \geq m_n(X) \geq w$. The generalized continuum hypothesis implies that either $m_n(X) = w$ $(= m(X))$ or else $m_n(X) = 2^w$. If $m_n(X) = 2^w$, then since $|C(X)| \geq m_n(X)$, $|C(X)| \geq 2^w$; since also $|C(X)| \leq 2^w$ by Lemma 3.9, it follows that $|C(X)| = 2^w = m_n(X)$.

3.11 *Theorem. If* X *is either the ordinal space* $[0,\omega_1]$ *or the one-point compactification of a discrete space of cardinality* $\aleph_1$, *then* $m_n(X) = \aleph_1$ *for all cardinal numbers* n.

Proof. Consider first the case $X = [0,\omega_1]$, where ω_1, as usual, denotes the smallest uncountable ordinal. For each $\xi \in [0,\omega_1)$, let $Y_\xi = X/[\xi,\omega_1]$ and let $p_\xi\colon X \to Y_\xi$ be the quotient map. Since Y_ξ is a countable compact Hausdorff space, it is embeddable in $\mathbf{R}$ [3, Theorem 5.40]; for each $\xi \in [0,\omega_1)$, let $j_\xi\colon Y_\xi \to \mathbf{R}$ be an embedding and let $F = \{j_\xi \circ p_\xi | \xi \in [0,\omega_1)\}$. We will show that $M^1(F) = C(X)$.

Let $h\colon X \to \mathbf{R}$ be any element of $C(X)$. It is well known, and easy to prove, that h is constant on some segment $[\xi,\omega_1]$, with $\xi \in [0,\omega_1)$. For each $y \in Y_\xi$, let $g(y) = h(p_\xi^{-1}(y))$; since $[\xi,\omega_1]$ is the only nondegenerate point-inverse of p_ξ and h is constant on $[\xi,\omega_1]$, g is a well defined function from Y_ξ into $\mathbf{R}$, and since $g \circ p_\xi = h$, it follows that g is continuous. Since $j_\xi\colon Y_\xi \to \mathbf{R}$ is an embedding, $g = \phi \circ j_\xi$ for some $\phi\colon \mathbf{R} \to \mathbf{R}$. Hence $\phi \circ j_\xi \circ p_\xi = g \circ p_\xi = h$ and since $j_\xi \circ p_\xi \in F$, it follows that $h \in M^1(F)$. Thus $C(X) \subset M^1(F)$, so $M^1(F) = C(X)$. Since $|F| = \aleph_1$, $m_1(X) \leq \aleph_1$.

By Theorem 3.2, X is embeddable in $I^{m(X)}$ and since the weight of X is $\aleph_1$, it follows that $m(X) \geq \aleph_1$. Thus for every cardinal number n, $\aleph_1 \geq m_1(X) \geq m_n(X) \geq m(X) \geq \aleph_1$, so $m_n(X) = \aleph_1$.

Now suppose X is the one-point compactification of a discrete space K, with $|K| = \aleph_1$. It is easy to show (e.g., by making K order isomorphic to $[0,\omega_1))$ that there is a set S of countable subsets of K with $|S| = \aleph_1$ such that every countable subset of K is contained in some element of S. For each $\sigma \in S$, $X - \sigma$ is compact; let $Y_\sigma = X/(X-\sigma)$, let $p_\sigma: X \to Y_\sigma$ be the quotient map and let $j_\sigma: Y \to \mathbf{R}$ be an embedding. Let $F = \{j_\sigma \circ p_\sigma | \sigma \in S\}$. Since every map $h: X \to \mathbf{R}$ is constant on the complement of a countable set, and therefore on the complement of some element of S, it follows exactly as before that $M^1(F) = C(X)$. Moreover, $|F| \leq |S| = \aleph_1$, so $m_1(X) \leq \aleph_1$. As before, this implies that $m_n(X) = \aleph_1$ for all n.

3.12 *Remark*. It can be shown by induction that for each positive integer k, if X is either the ordinal space $[0,\omega_k]$ or the one-point compactification of a discrete space of cardinality $\aleph_k$, then $m_n(X) = \aleph_k$ for all n. We have been unable to obtain similar results for larger alephs. Construction of such examples might lead to an equivalence for GCH similar to the one for CH given in the next theorem.

3.13 *Theorem*. *The continuum hypothesis is equivalent to the following proposition: For every compact Hausdorff space* X*, either* $m_1(X) = 1$ *or* $m_1(X) = |C(X)|$.

Proof. Assuming the continuum hypothesis, we have, by the proof of Theorem 3.8, that $m_1(X) = |C(X)|$ whenever $m_1(X)$ is uncountable; on the other hand, if $m_1(X)$ is countable, then by Theorem 3.6, $m_1(X) \leq 1$ and thus $m_1(X) = 1$.

Conversely, applying the given proposition to either of the examples of Theorem 3.11 gives $\aleph_1 = |C(X)|$ and since $|C(X)| \geq \mathfrak{c}$ for any X, we have $\aleph_1 \geq \mathfrak{c}$ and hence $\aleph_1 = \mathfrak{c}$.

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