CONTINUOUS THAT ARE LOCALLY A BUNDLE OF ARCS

by

ANDRZEJ GUTEK AND JAN VAN MILL
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A continuum is a compact connected Hausdorff space.
A continuum is decomposable if it can be represented as a union of two of its proper subcontinua; otherwise, it is indecomposable.

A continuum is locally a bundle of arcs if there exists a compact totally disconnected space $X$ such that every point has a neighbourhood homeomorphic to $X \times (0,1)$.

It is rather easy to construct continua that are locally a bundle of arcs. Let $X$ be a space consisting of two sequences with limit points, say

$$X = \{a_n : n \text{ is an integer} \} \cup \{0,3\},$$

where $a_n = \frac{1}{n}$ if $n$ is a positive integer and

$$a_n = 3 + \frac{1}{n-1} \text{ if } n \text{ is a negative integer or zero.}$$

Let $f$ be a homeomorphism from $X$ onto itself defined by

$$f(0) = 0,$$

$$f(a_n) = a_{n+1},$$

$$f(3) = 3.$$ 

The space $X \times I/f$ obtained from the product $X \times I$, where $I$ is the closed unit interval, by identifying for each $x \in X$ the point $(x,0)$ with $(f(x),1)$ is a planar continuum (see the picture).
Remark 1. There exists in the plane an indecomposable continuum that is locally homeomorphic to $C \times (0,1)$, where $C$ denotes the Cantor set.

In Bing's paper [1], on pages 222 and 223 a description is given of an example of such an indecomposable plane continuum $Y$. Every proper subcontinuum of $Y$ is an arc and $Y$ is locally homogeneous (i.e., for each pair of points $p, q$ of $Y$ there are arbitrarily small homeomorphic open subsets $N_p, N_q$ containing $p, q$, respectively). Hence, $Y$ is also locally a bundle $C \times (0,1)$.

We are indebted to F. Burton Jones for referring us to this example.

The next two theorems describe continua that are locally a bundle of arcs.

Theorem 1 [3, p. 29]. Let $X$ be a compact totally disconnected Hausdorff space. If a continuum $K$ is locally a bundle $X \times (0,1)$, then $K$ can be obtained as a quotient of the product $X \times I$ by identifying for each $x \in X$ the points $(x,e)$ and $h(x,e)$, where $h$ is an involution with no fixed points defined on $X \times \{0,1\}$. 
Theorem 2 [2, Corollary on page 552]. Let \( X \) be a compact Hausdorff totally disconnected and dense in itself space. If a homeomorphism \( f \) from \( X \) onto itself is such that for some \( x \) in \( X \) the set \( \{ f^n(x) : n \text{ is an integer} \} \) is dense in \( X \), then the space \( X \times I/f \) obtained from the product \( X \times I \) by identifying for each \( x \in X \) the points \( (x,0) \) with \( (f(x),1) \), is an indecomposable continuum.

To insure the existence of homeomorphisms described in the preceding theorem, we use the following:

Theorem 3. Let \( P \) and \( Q \) be closed and nowhere dense subsets of the Cantor set \( C \), and let \( h \) be a homeomorphism from \( P \) onto \( Q \). Then there exists an extension \( h' \) of \( h \) such that \( h' \) is a homeomorphism from \( C \) onto itself while moreover for some \( c \) of \( C \) the set \( \{(h')^n(c) : n \text{ is an integer} \} \) is a dense subset of \( C \).

The first proof of this theorem was published in [4]. Based on the well-known idea of a shift-mapping, we give a short proof of the theorem.

Proof. Let us observe that because \( P \cup Q \) is a nowhere dense subset of \( C \), then there exists a nowhere dense closed subset \( D \) of \( C \), which is homeomorphic to \( C \) and which is such that \( P \cup Q \) is nowhere dense in \( D \). By the theorem of Ryll-Nardzewski [7, Corollary 2, p. 186] there exists a homeomorphism from \( D \) onto itself that is an extension of \( h \).

Thus, without the loss of generality, we can assume that \( P = Q \)
Let $\mathbb{Z}$ denote the set of integers and let $\mathbb{C}^\mathbb{Z}$ be the product of countably many copies of the Cantor set.

Let $i_p$ be an embedding of the set $\mathbb{P}$ into $\mathbb{C}^\mathbb{Z}$ defined by

$$i_p(p) = (\cdots, h^{-1}(p), p, h(p), h^2(p), \cdots),$$

where $p$ is the zero-coordinate, $h(p)$ is the 1-coordinate of $i_p(p)$, and so on.

Because we assume that $\mathbb{P} \subseteq \mathbb{Q}$, the function $i_p$ is also an embedding of $\mathbb{Q}$.

Let $s$ be the shift-mapping from $\mathbb{C}^\mathbb{Z}$ onto itself defined by $s((c_n)_{n\in\mathbb{Z}}) = (c_{n+1})_{n\in\mathbb{Z}}$.

Observe that $s(i_p(p)) = i_p(h(p))$. By the already mentioned theorem of Ryll-Nardzewski there exists a homeomorphism $f_p$ from $\mathbb{C}$ onto $\mathbb{C}^\mathbb{Z}$ such that $f_p|\mathbb{P} = i_p$ (see the diagram).

The homeomorphism $h' = f_p^{-1} \cdot s \cdot f_p$ is an extension of $h$.

Let $\mathbb{A}$ be a countable dense subset of $\mathbb{C}^\mathbb{Z}$, i.e.,

$$\mathbb{A} = \{(d_n^i)_{n\in\mathbb{Z}}: i \text{ is a positive integer}\}.$$  

Let $d$ be a point of $\mathbb{C}^\mathbb{Z}$ such that for any finite subset $(x_1, \cdots, x_m)$ of $\{d_n^i: i \text{ is a positive integer and } n \in \mathbb{Z}\}$ there is a $k \in \mathbb{Z}$ such that $d_{k+j} = x_j$ for all $1 \leq j \leq m$.

The orbit $\{(h')^k(f_p^{-1}(d)): k \text{ is an integer}\}$ is dense in $\mathbb{C}$.
**Question 1.** Does there exist a planar indecomposable continuum obtained from $C \times I$ by identifying for each $x \in C$ the point $(x,0)$ with $(f(x),1)$, for some homeomorphism from $C$ onto itself?

Ch. L. Hagopian proved in [6] that an indecomposable continuum is a solenoid if and only if it is homogeneous and all of its proper subcontinua are arcs.

R. D. Anderson pointed out that there is a continuum on the surface of a torus that is indecomposable, has only arcs as proper subcontinua, is locally a bundle $C \times (0,1)$ and is obtained from the product $C \times I$ by identifying for each $x \in C$ the point $(x,0)$ with $(f(x),1)$, for some homeomorphism $f$ from the Cantor set $C$ onto itself. This continuum is not a solenoid.

The Anderson example can be described in the following way.

Let $C = \{(e^{it}: t \in \mathbb{R}) \times \{0\}\} \cup \{(e^{in}: n \text{ is an integer}) \times \{1\}\}$, where $i = \sqrt{-1}$. Induce a topology on $C$ by taking the following sets as a basis: $\{\{(e^{it}: r \leq t < s) \times \{0,1\}\} \cup \{(e^{is},1)\}\} \cap C$, where $r$ and $s$ are reals such that $e^{ir} = e^{in}$ and $e^{is} = e^{ik}$ for some integers $n$ and $k$. It is easy to see that $C$ with this topology is the Cantor set. Define a homeomorphism $f$ from $C$ onto itself by $f(e^{it},j) = (e^{i(t+1)},j)$. The orbit of every point is dense and the space $C \times I/f$ is an indecomposable continuum.

**Question 2.** Let $M$ be the family of all continua that are indecomposable, locally a bundle $C \times (0,1)$, have only
arcs as proper subcontinua, are not solenoids and are obtained from $C \times I$ by identifying $C \times \{0\}$ with $C \times \{1\}$ under some homeomorphism. Does the family $\mathcal{H}$ contain $2^\omega$ non-homeomorphic continua?

**Question 3.** In [5], an example was found of a homogeneous indecomposable and non-metrizable continuum that has only arcs as proper subcontinua. Are there $2^\omega$ non-homeomorphic continua of this type?

**Question 4.** Let $K$ be a homogeneous indecomposable circle-like continuum that has only arcs as proper subcontinua. Is the continuum $K$ metrizable (and hence a solenoid)?

**References**


California State University
Sacramento, California 95819

and

Vrije Universiteit
Subfaculteit Wiskunde
De Boelelaan 1081
Amsterdam, Holland