C-SETS AND MAPPINGS OF CONTINUA

by

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1. Introduction

In 1955 A. D. Wallace [9] introduced the study of C-sets and investigated C-sets in semigroups. In this paper we investigate C-sets in Hausdorff continua (compact, connected Hausdorff spaces) and note some properties of C-sets pertaining to the study of mappings onto continua. If M is a continuum, a proper subset H of M is a C-set in M provided H is a subset of any subcontinuum of M which contains both a point in H and a point not in H. In Lemma 1 of [9, p. 639] Wallace observed that C-sets are connected and have no interior. Although C-sets do not have to be closed, it is not difficult to show that if K is a C-set which is not closed then K is an indecomposable continuum. Moreover, if K is a C-set which is not closed the K is the union of some of the composants of K. To see this suppose H is a subcontinuum of K containing a point in K and a point not in K. Then H contains K and thus H contains K, so each proper subcontinuum of K which intersects K is a subset of K. Consequently, each composant of K which intersects K is a subset of K.

A continuum M is a triod provided it contains a subcontinuum C such that M-C has at least three components. A continuum is said to be atriodic provided it contains no triod. The statement that the continuum M is unicoherent
means if A and B are continua whose union is M then \( A \cap B \) is connected. In his doctoral dissertation at the University of Houston, Collins [2] introduced the class of IUC continua and proved [2, Theorem 6, p. 12] that atriodic continua have property IUC hereditarily. A continuum has property IUC provided every proper subcontinuum with interior is unicoherent. Collins' result that atriodic continua have property IUC has been obtained independently by Mackowiak and Tymchatyn [7]. In this paper we generalize these results (Theorem 3).

The so-called "boundary bumping theorem" is used often in the proofs in this paper. For a proof of it in Hausdorff continua see [5, Theorem 2, p. 172].

2. A Characterization of C-Sets

The following theorem, although not stated in this manner, is essentially what Cook [1, Theorem 4, p. 243] and Read [8] (see also [6, 5.7, p. 111]) proved when they showed that a continuum is hereditarily indecomposable if and only if every mapping of a continuum onto it is confluent. The proof presented here differs only slightly and is included only for the sake of completeness.

**Theorem 1.** Suppose \( M \) is a Hausdorff continuum and \( H \) is a proper subcontinuum of \( M \). Then \( H \) is a C-set in \( M \) if and only if for each mapping \( f \) of a continuum onto \( M \) every component of \( f^{-1}(H) \) is thrown by \( f \) onto \( H \).

**Proof.** Suppose \( M \) is a continuum, \( H \) is a subcontinuum of \( M \) which is a C-set and \( f \) is a mapping of a continuum \( X \)
onto M. Let K be a component of $f^{-1}(H)$ and G be a monotonic collection of subcontinua of X such that the common part of all the members of G is K and each member of G contains a point not in K. Then, if J is a continuum in G, $f[J]$ contains a point of H and a point not in H, so H is a subset of $f[J]$. Since X is a Hausdorff continuum, G is monotonic and K is the common part of all the members of G, $f[K] = \bigcap_{J \in G} f[J]$. Thus, $f[K] = H$.

On the other hand suppose H is not a C-set and C is a subcontinuum of M not containing H which contains a point of H and a point Q not in H. Let X be the continuum obtained by identifying $(Q,0)$ and $(Q,1)$ in $(M \times \{0\}) \cup (C \times \{1\})$ and $f$ be the natural projection of X onto M. Then $f^{-1}(H)$ has two components one of which is not thrown onto H by f. This completes the proof of Theorem 1.

Remark. It is easy to show that a continuum is hereditarily indecomposable if and only if every proper subcontinuum of it is a C-set in it.

3. Atriodic Continua and C-Sets

In this section we often use the following property of atriodic continua: If M is a decomposable, atriodic continuum then M is the union of two continua A and B such that $A = \overline{A - (A \cap B)}$ and $B = \overline{B - (A \cap B)}$. For a proof of this see Collins [2] or [3]. It should be noted that in Collins' work he assumes that continua are metric but his arguments do not require changes for Hausdorff continua.
Lemma. If \( A \) and \( B \) are two continua which intersect such that (1) \( A \cup B \) is atriodic, (2) \( A = \overline{A - (A \cap B)} \) and \( B = \overline{B - (A \cap B)} \), and (3) \( A \cap B \) is the union of the two continua \( C_1 \) and \( C_2 \), then \( A \) is irreducible from \( C_1 \) to \( C_2 \).

Proof. Suppose \( P \) is a proper subcontinuum of \( A \) which intersects both \( C_1 \) and \( C_2 \) and let \( y \) be a point of \( A \) not in \( P \cup C_1 \cup C_2 \). That there is such a point \( y \) follows by the assumption that \( A = \overline{A - (A \cap B)} \) for if \( P \) contains \( A - (C_1 \cup C_2) \) then \( P \) contains \( A \). There exist mutually exclusive open sets \( U_1 \) and \( U_2 \) containing \( C_1 \) and \( C_2 \) respectively such that \( \overline{U_1} \) and \( \overline{U_2} \) are mutually exclusive and neither contains \( y \). Let \( B_1 \) and \( B_2 \) be the components of \( B \cap U_1 \) and \( B \cap U_2 \) containing \( C_1 \) and \( C_2 \) respectively. Then \( A \cup (P \cup B_1) \cup (P \cup B_2) \) is a triod.

The following theorem was proved independently by Maćkowiak and Tymchatyn [7, 13(2), p. 40]. In that paper they call C-sets which are continua terminal continua. This theorem is generalized in the next section of this paper.

Theorem 2. If \( M \) is an atriodic continuum then each proper subcontinuum of \( M \) which is not unicoherent is a C-set.

Proof. Suppose \( H \) is a proper subcontinuum of \( M \) such that \( H \) is not unicoherent and \( H \) is not a C-set. Then \( H \) is the union of two continua \( A \) and \( B \) such that \( A \cap B \) is not connected and \( A = \overline{A - (A \cap B)} \) and \( B = \overline{B - (A \cap B)} \). Suppose \( K \) is a subcontinuum of \( M \) containing a point of \( A \cup B \) and a
point not in \( A \cup B \). Since \( A \cap B \) is not connected and \( M \) is atriodic, \( A \cap B \) is the union of two continua \( C_1 \) and \( C_2 \).

Suppose \( K \) contains a point of \( B \). We now show that \( K \) contains \( A \).

Suppose \( x \) is a point of \( A - (A \cap B) \) which is not in \( K \). There is an open set \( U \) containing \( x \) which does not contain a point of \( B \cup K \). By the Lemma, \( A \) is irreducible from \( C_1 \) to \( C_2 \) so \( A - (A \cap U) \) contains no continuum intersecting both \( C_1 \) and \( C_2 \). Therefore, [5, Theorem 1, p. 168], \( A - (A \cap U) \) is the union of two mutually exclusive closed point sets \( H_1 \) and \( H_2 \) containing \( C_1 \) and \( C_2 \) respectively.

There exist mutually exclusive open sets \( U_1 \) and \( U_2 \) containing \( H_1 \) and \( H_2 \) respectively such that \( \overline{U}_1 \) and \( \overline{U}_2 \) are mutually exclusive and neither contains \( x \). Let \( A_1 \) and \( A_2 \) denote the components of \( A \cap U_1 \) and \( A \cap U_2 \) containing \( C_1 \) and \( C_2 \) respectively. Then \( (B \cup K) \cup (B \cup \overline{A}_1) \cup (B \cup \overline{A}_2) \) is a triod.

Now, since \( K \) contains \( A \), by repeating the argument of the previous paragraph exchanging the roles of \( A \) and \( B \), we obtain the \( K \) contains \( B \). This will complete the proof.

4. HIUC Continua and C-Sets

A continuum having property IUC hereditarily is said to have property HIUC.

**Theorem 3.** If \( M \) is a continuum with property HIUC then each proper subcontinuum of \( M \) which is not unicoherent is a C-set.

**Proof.** Suppose \( H \) is a non-unicoherent proper subcontinuum of \( M \), and \( K \) is a subcontinuum of \( M \) containing a
point of $H$ and a point not in $H$. Suppose $x$ is a point of $H$ which is not in $K$. Then there is an open set $U$ containing $x$ which contains no point of $K$. Then $H \cup K$ does not have property IUC since $H$ is a non-unicoherent proper subcontinuum of $H \cup K$ which has interior in $H \cup K$.

Remark. It is easy to see from the example below that the hypothesis in Theorem 3 that $M$ have property HIUC may not be weakened to $M$ has property IUC for the circle is not a C-set in $M$.

5. Confluence and Weak Confluence

We conclude this paper with some consequences of Theorems 1, 2 and 3. First, we introduce some terminology which the author has found useful in discussing confluence and related properties.

Definitions. Suppose $M$ is a continuum, $H$ is a subcontinuum of $M$ and $f$ is a mapping of a continuum onto $M$. The statement that $f$ is confluent with respect to $H$ (respectively, weakly confluent with respect to $H$) means each (resp., some) component of $f^{-1}(H)$ is thrown by $f$ onto $H$.

Thus, if $f$ is a mapping of a continuum onto $M$ then $f$ is confluent (resp., weakly confluent) provided $f$ is confluent (resp., weakly confluent) with respect to each
non-degenerate proper subcontinuum of M. Further, f is said to be pseudo-confluent provided f is weakly confluent with respect to each irreducible subcontinuum of M.

Theorem 1 may now be restated: A proper subcontinuum H of a continuum M is a C-set in M if and only if every mapping of a continuum onto M is confluent with respect to H.

The following theorems are immediate from Theorems 1 and 3.

Theorem 4. Suppose f is a mapping of a continuum onto the continuum M and M has property HIUC. If H is a non-unicoherent proper subcontinuum of M then f is confluent with respect to H.

Theorem 5. If f is a mapping of a continuum onto a continuum M having property HIUC then f is confluent (resp., weakly confluent) if and only if f is confluent (resp., weakly confluent) with respect to every unicoherent proper subcontinuum of M.

Corollary 1. Suppose f is a mapping of a continuum onto an atriodic continuum. Then f is pseudo-confluent if and only if f is weakly confluent.

Finally, we observe that Corollary 1 provides another proof of a theorem of Grispolakis and Tymchatyn [4, Theorem 5.3].

Corollary 2. Suppose M is an atriodic continuum. Then M is in Class W if and only if M is in Class P.
References

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