Research Announcement:

A NOTE ON GALE’S PROPERTY (G)

by

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In [1], Gale gave a condition which could be used to replace equicontinuity in a less restrictive version of Ascoli's Theorem, namely where the range space is regular, rather than a metric or uniform space. Gale's Theorem 1 is stated below for the sake of completeness. Throughout this note, $Y^X$ will denote the collection of all functions from $X$ to $Y$ with the product topology and if $F \subset Y^X$, then $\overline{F}$ will denote the closure of $F$ in this topology, i.e., the pointwise closure of $F$.

**Theorem 1 (Gale).** If $X$ is a k-space and $Y$ is regular, then a collection of continuous functions $F$ from $X$ to $Y$ is compact in the compact-open topology if and only if

1. $F$ is closed.
2. $F(x)$ is compact for each $x$ in $X$.
3. If $G$ is closed in $F$ and $U$ is open in $Y$ then $\cap \{g^{-1}(U) | g \in G\}$ is open in $X$.

In the proof of this theorem, Gale showed that if a collection $F$ is continuous and satisfies condition (3) then the compact-open and pointwise topologies agree on $F$. This condition was abstracted by Yang in [5] and renamed property (G).

**Definition.** $F \subset Y^X$ is said to have property (G) if for each $U$ open in $Y$, and each pointwise closed subset $G$ of $F$, $\cap \{g^{-1}(U) | g \in G\}$ is open in $X$. 
Note that in the above definition, the topology being considered is the pointwise, rather than the compact-open, and F is not required to be a closed collection, as was the case in Gale’s Theorem 1. Since F is not necessarily closed, the phrase "pointwise closed subset G of F" admits two distinct interpretations. Either

1. the closure of G in \( Y^X \) lies in F, or
2. G is closed in the relative topology on F induced by \( Y^X \).

The purpose of this note is to examine this ambiguity.

Kelley, to whom Yang refers for all definitions not specified in [5], defines pointwise closed [4, p. 218] to mean closed in \( Y^X \), so that interpretation (1) of property (G) seems to be intended. Yet the proof of Theorem 1 of [5] employs interpretation (2), and in fact is false using interpretation (1), as our Example B will show.

In order to sort out these difficulties, we will introduce two versions of the definition of Property (G).

Using interpretation (1) we will say \( F \subseteq Y^X \) has property (\( G_1 \)) if for each open U in Y, and for each \( G \subseteq F \) such that \( \overline{G} = G \), \( \cap \{ g^{-1}(U) \mid g \in G \} \) is open in X.

Similarly we will say \( F \subseteq Y^X \) has property (\( G_2 \)) if for each open U of Y and for each \( G \subseteq F \) such that \( G = \overline{G} \cap F \), \( \cap \{ g^{-1}(U) \mid g \in G \} \) is open in X.

It is clear that if F satisfies property (\( G_2 \)) then F must also satisfy property (\( G_1 \)), but the converse fails as the following example shows.
Example A. For each $n \in \mathbb{N}$, define $f_n : [0,1] \to [0,1]$ by

$$f_n(x) = \begin{cases} 
\frac{1}{2n} & , \quad x \in [0,1/2n] \\
x & , \quad x \in [1/2n,1]
\end{cases}$$

and let $F = \{f_n | n \in \mathbb{N}\}$. Then $F$ has property $(G_1)$ trivially because the only subsets $G$ of $F$ for which $\overline{G} \subset F$ are the finite ones, so the intersection condition is always satisfied. But letting $U_0 = (0,1) - \{1/(2n+1) | n \in \mathbb{N}\}$ and noting that $F \cap F$, we have that

$$\cap \{f_n^{-1}(U_0) | n \in \mathbb{N}\} = [0,1) - \{1/(2n+1) | n \in \mathbb{N}\}$$

which is not open in $X$, so that $F$ does not satisfy property $(G_2)$. Also note that $F$ is equicontinuous and pointwise bounded, and therefore regular by the corollary to Theorem 3 of [5].

The proof of Theorem 1 of [5] establishes that a collection satisfying property $(G_2)$ is necessarily regular, but the next example shows that this result fails for property $(G_1)$.

Example B. For each $n \in \mathbb{N}$, define $f_n : [0,1] \to [0,1]$ by

$$f_n(x) = \begin{cases} 
4nx & , \quad x \in [0,1/4n] \\
2-4nx & , \quad x \in [1/4n,1/2n] \\
0 & , \quad x \in [1/2n,1]
\end{cases}$$

and let $F = \{f_n | n \in \mathbb{N}\}$. Then $F$ has property $(G_1)$ trivially, but is not equicontinuous at $x = 0$, and hence by Theorem 5 of [2] is not regular there.
Examples A and B also show that the corollary following Theorem 6 of [5] fails under either interpretation of property (G). However it is the case that whenever $X$ is a k-space and $Y$ is regular, if $F$ is evenly continuous (or regular, by Theorem A of [3]) and $\overline{F(x)}$ is compact for each $x$ in $X$, then $F$ satisfies property $(G_1)$. This holds because if $F$ is evenly continuous, then so is $\overline{F}$ by [4, Theorem 19, p. 235], and hence the product topology and the compact-open topology coincide on $\overline{F}$. Thus $\overline{F}$ is compact by [6, Theorem B] and therefore satisfies property $(G_1)$ by Gale's Theorem 1. It follows from the definition that $F$ must also satisfy property $(G_1)$.

References


West Virginia University
Morgantown, West Virginia 26506