Research Announcement:

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by

M. Henry, D. Reynolds and G. Trapp

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Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
      Department of Mathematics & Statistics
      Auburn University, Alabama 36849, USA
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In [1], Gale gave a condition which could be used to replace equicontinuity in a less restrictive version of Ascoli's Theorem, namely where the range space is regular, rather than a metric or uniform space. Gale's Theorem 1 is stated below for the sake of completeness. Throughout this note, \( Y^X \) will denote the collection of all functions from \( X \) to \( Y \) with the product topology and if \( F \subset Y^X \), then \( \overline{F} \) will denote the closure of \( F \) in this topology, i.e., the pointwise closure of \( F \).

**Theorem 1 (Gale).** If \( X \) is a k-space and \( Y \) is regular, then a collection of continuous functions \( F \) from \( X \) to \( Y \) is compact in the compact-open topology if and only if

1. \( F \) is closed.
2. \( F(x) \) is compact for each \( x \) in \( X \).
3. If \( G \) is closed in \( F \) and \( U \) is open in \( Y \) then \( \cap \{ g^{-1}(U) \mid g \in G \} \) is open in \( X \).

In the proof of this theorem, Gale showed that if a collection \( F \) is continuous and satisfies condition (3) then the compact-open and pointwise topologies agree on \( F \). This condition was abstracted by Yang in [5] and renamed property (G).

**Definition.** \( F \subset Y^X \) is said to have property (G) if for each \( U \) open in \( Y \), and each pointwise closed subset \( G \) of \( F \), \( \cap \{ g^{-1}(U) \mid g \in G \} \) is open in \( X \).
Note that in the above definition, the topology being considered is the pointwise, rather than the compact-open, and F is not required to be a closed collection, as was the case in Gale's Theorem 1. Since F is not necessarily closed, the phrase "pointwise closed subset G of F" admits two distinct interpretations. Either

1. the closure of G in $Y^X$ lies in F, or
2. G is closed in the relative topology on F induced by $Y^X$.

The purpose of this note is to examine this ambiguity.

Kelley, to whom Yang refers for all definitions not specified in [5], defines pointwise closed [4, p. 218] to mean closed in $Y^X$, so that interpretation (1) of property (G) seems to be intended. Yet the proof of Theorem 1 of [5] employs interpretation (2), and in fact is false using interpretation (1), as our Example B will show.

In order to sort out these difficulties, we will introduce two versions of the definition of Property (G).

Using interpretation (1) we will say $F \subseteq Y^X$ has property $(G_1)$ if for each open $U$ in $Y$, and for each $G \subseteq F$ such that $\overline{G} = G$, $\cap \{g^{-1}(U) | g \in G\}$ is open in $X$.

Similarly we will say $F \subseteq Y^X$ has property $(G_2)$ if for each open $U$ of $Y$ and for each $G \subseteq F$ such that $G = \overline{G} \cap F$, $\cap \{g^{-1}(U) | g \in G\}$ is open in $X$.

It is clear that if $F$ satisfies property $(G_2)$ then $F$ must also satisfy property $(G_1)$, but the converse fails as the following example shows.
Example A. For each \( n \in \mathbb{N} \), define \( f_n : [0,1] \to [0,1] \) by

\[
  f_n(x) = \begin{cases} 
    1/2n, & x \in [0,1/2n] \\
    x, & x \in [1/2n,1]
  \end{cases}
\]

and let \( F = \{f_n | n \in \mathbb{N}\} \). Then \( F \) has property \((G_1)\) trivially because the only subsets \( G \) of \( F \) for which \( \overline{G} \subset F \) are the finite ones, so the intersection condition is always satisfied. But letting \( U_0 = (0,1) - \{1/(2n+1) | n \in \mathbb{N}\} \) and noting that \( F \subseteq \overline{F} \cap F \), we have that

\[
  \cap \{f_n^{-1}(U_0) | n \in \mathbb{N}\} = [0,1) - \{1/(2n+1) | n \in \mathbb{N}\}
\]

which is not open in \( X \), so that \( F \) does not satisfy property \((G_2)\). Also note that \( F \) is equicontinuous and pointwise bounded, and therefore regular by the corollary to Theorem 3 of [5].

The proof of Theorem 1 of [5] establishes that a collection satisfying property \((G_2)\) is necessarily regular, but the next example shows that this result fails for property \((G_1)\).

Example B. For each \( n \in \mathbb{N} \), define \( f_n : [0,1] \to [0,1] \) by

\[
  f_n(x) = \begin{cases} 
    4nx, & x \in [0,1/4n] \\
    2-4nx, & x \in [1/4n,1/2n] \\
    0, & x \in [1/2n,1]
  \end{cases}
\]

and let \( F = \{f_n | n \in \mathbb{N}\} \). Then \( F \) has property \((G_1)\) trivially, but is not equicontinuous at \( x = 0 \), and hence by Theorem 5 of [2] is not regular there.
Examples A and B also show that the corollary following Theorem 6 of [5] fails under either interpretation of property (G). However it is the case that whenever X is a k-space and Y is regular, if F is evenly continuous (or regular, by Theorem A of [3]) and $F(x)$ is compact for each $x$ in $X$, then $F$ satisfies property $(G_1)$. This holds because if $F$ is evenly continuous, then so is $F$ by [4, Theorem 19, p. 235], and hence the product topology and the compact-open topology coincide on $F$. Thus $F$ is compact by [6, Theorem B] and therefore satisfies property $(G_1)$ by Gale's Theorem 1. It follows from the definition that $F$ must also satisfy property $(G_1)$.

References