CLOSED IMAGES OF LOCALLY COMPACT SPACES AND FRECHET SPACES

by
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0. Introduction

Every quotient image of a metric space is actually the quotient image of a locally compact metric space [4]. However, every closed image (closed s-image) of a metric space need not be the closed image (closed s-image) of a locally compact metric space. Indeed, any non-locally compact, metric space is not the closed image of a locally compact metric space.

So, the for image $Y$ of a metric space (paracompact $M$-space) under a closed map, a closed $s$-map, or a pseudo-open $s$-map, we give a necessary and sufficient condition for $Y$ to be the image of a locally compact metric space (locally compact paracompact space) under the respective kind of map. Also, we show that every Fréchet space which is the quotient $s$-image of a locally compact metric space is a Lašnev space which can be decomposed into a discrete closed subspace and a locally compact subspace.

We assume that all spaces are regular; all maps are continuous and onto.

1. Closed Images of Locally Compact Paracompact Spaces

In [9], E. Michael introduced the notion of bi-$k$-spaces and showed that every bi-$k$-space is precisely the bi-quotient image of a paracompact $M$-space. Hence, every paracompact
M-space is bi-k. Recall that a space is a paracompact M-space if it admits a perfect map onto a metric space.

A space $X$ is a bi-k-space [9; Lemma 3.E.2], if whenever $F$ is a filter base accumulating at $x$ in $X$, then there exists a $k$-sequence $(F_n)$ in $X$ such that $x \in \overline{F \cap F_n}$ for all $F_n$ and all $F \in J$. Here, a decreasing sequence $(F_n)$ of subsets of $X$ is a $k$-sequence, if $K = \bigcap_{n=1}^{\infty} F_n$ is compact in $X$ and every neighborhood of $K$ contains some $F_n$. We can assume that all $F_n$ are closed in $X$.

**Theorem 1.1.** The following are equivalent.

1. $Y$ is the closed image of a paracompact bi-k-space, and each closed (or closed $\sigma$-compact) M-subspace of $Y$ is locally compact.
2. $Y$ is the closed image of a locally compact paracompact space.

**Proof.** (2) $\Rightarrow$ (1): Let $f$ be a closed map from a locally compact paracompact space onto $Y$. Let $Y_1$ be a closed, M-subspace of $Y$, and $X_1 = f^{-1}(Y_1)$. Then $g = f|X_1$ is a closed map from a paracompact space $X_1$ onto an M-space $Y_1$. Thus, each $g^{-1}(y)$ is compact by [7; Corollary 2.2]. Hence, as in the proof of [7; Corollary 1.2], there is a closed subset $F$ of $X_1$ such that $g|F$ is a perfect map onto $Y_1$. Since $F$ is locally compact, so is $Y_1$. Hence each closed, M-subspace of $Y$ is locally compact.

(1) $\Rightarrow$ (2): Let $f: X \rightarrow Y$ be a closed map with $X$ paracompact bi-k. Let $y \in Y$. Then we will prove that each point of $f^{-1}(y)$ has a neighborhood contained in the inverse image of some compact subset of $Y$. To see this, suppose
not. Then there is a point \( x_0 \in f^{-1}(y) \) such that for any neighborhood \( V \) of \( x_0 \) and for any compact subset \( K \) of \( Y \), \( V \notin f^{-1}(K) \). Let \( J = \{ X - f^{-1}(K); K \text{ is compact in } Y \} \).

Then \( J \) is a filter base accumulating at \( x_0 \). Since \( X \) is a bi-k-space, there is a k-sequence \( (F_n) \) in \( X \) with \( x_0 \in F_n \cap F \) for all \( F_n \) and all \( F \in J \). Obviously, \( (f(F_n)) \) is a k-sequence in \( Y \). Suppose that all \( f(F_n) \) are not compact. Now, recall the well-known result due to E. Michael: Every closed image of a paracompact space is paracompact (see, [2; Theorem 2.4, p. 165]). Thus \( Y \) is paracompact. Then, the \( f(F_n)'s \) are not countably compact. Hence there are closed, countable discrete subsets \( D_n \) of \( f(F_n) \). Let \( C = \cap_{n=1}^{\infty} f(F_n) \) and \( Y_0 = C \cup \cup_{n=1}^{\infty} D_n \). Then \( Y_0 \) is closed in \( Y \). Let \( Z \) be the quotient space obtained from \( Y_0 \) by identifying the compact subset \( C \) to a point. Then it is easy to show that \( Y_0 \) is the perfect pre-image of a countable metric space \( Z \) and that \( Z \) is not locally compact. Thus, \( Y_0 \) is a closed, \( \sigma \)-compact \( M \)-subspace of \( Y \) which is not locally compact. This is a contradiction to the hypothesis of \( Y \). Hence some \( f(F_{n_0}) \) is compact. Let \( K_0 = f(F_{n_0}) \). Then \( K_0 \) is compact in \( Y \). But,

\[
x_0 \in (X - f^{-1}(K_0)) \cap F_{n_0} \subset (X - f^{-1}(K_0)) \cap f^{-1}(K_0) = \emptyset.
\]

This contradiction implies that each point of \( f^{-1}(y) \) has a neighborhood which is contained in the inverse image of some compact subset of \( Y \). Let \( \mathcal{V} = \{ V; V \text{ is open in } X \text{ with } V \subset f^{-1}(K) \text{ for some compact } K \subset Y \} \). Then \( \mathcal{V} \) is an open covering of a paracompact space \( X \). Thus \( \mathcal{V} \) has a locally finite closed refinement \( \mathcal{C} \). Let \( f(\mathcal{C}) = \{ f(C); C \in \mathcal{C} \} \). Then, since \( f \) is closed, \( f(\mathcal{C}) \) is a hereditarily closure-preserving
cover of compact subsets of $Y$. Let $Z$ be the topological sum of elements of $f(\cdot)$, and let $g: Z \to Y$ be the obvious map. Then $g$ is a closed map from a locally compact paracompact space $Z$ onto $Y$.

We shall call a map Lindelöf if every point-inverse is a Lindelöf space.

**Corollary 1.2.** The following are equivalent.

1. $Y$ is the closed Lindelöf image of a bi-k-space, and $Y$ is a paracompact space in which every closed (or closed $\sigma$-compact) $M$-subspace is locally compact.

2. $Y$ is the closed Lindelöf image of a locally compact paracompact space.

**Proof.** It suffices to prove (1) $\Rightarrow$ (2). Let $f: X \to Y$ be a closed Lindelöf map with $X$ bi-k. Let $y \in Y$. Then, by the proof of (1) $\Rightarrow$ (2) of the previous theorem, there is a sequence $\{V_n; n \in \mathbb{N}\}$ of open subsets of $X$ such that $f^{-1}(y) \subseteq \bigcup_{n=1}^{\infty} V_n$ and each $f(V_n)$ is compact. Since $f$ is closed, $y \in \text{int}(\bigcup_{n=1}^{\infty} f(V_n))$. This shows that $Y$ is locally $\sigma$-compact. Since $Y$ is paracompact, there is a locally finite closed cover of $\sigma$-compact subspaces $F_\alpha$. Since $f$ is a closed Lindelöf map, each $f^{-1}(F_\alpha)$ is Lindelöf. Thus each $F_\alpha$ is the closed image of a Lindelöf (hence paracompact) bi-k-space. Thus, by the proof of the previous theorem, each $F_\alpha$ has a countable hereditarily closure-preserving cover of compact subsets. So, each $F_\alpha$ is the closed image of a locally compact, Lindelöf space. Hence $Y$ is the closed Lindelöf image of a locally compact paracompact space. That completes the proof.
Let $Y$ be the closed image of a paracompact space $X$. Then every compact subset of $Y$ is the image of some compact subset of $X$ [7; Corollary 1.2]. So, in view of the proof of Theorem 1.1 and Corollary 1.2, we have

**Corollary 1.3.** Let $Y$ be the closed image (resp. closed Lindelöf image) of a paracompact bi-k-space $X$. Then $Y$ is the closed image (resp. closed Lindelöf image) of the topological sum of some compact subsets of $X$ if and only if every closed (c-compact) M-subspace of $Y$ is locally compact.

Now we shall consider Lašnev spaces. Recall that a space is Lašnev if it is the closed image of a metric space. Let $f: X \to Y$ be a closed map with $X$ metric. Then for each $x \in X$ and for each decreasing local base $\{V_n; n \in \mathbb{N}\}$ at $x$, $(f(V_n))$ is a k-sequence with $\cap_{n=1}^{\infty} f(V_n)$ a single point $f(x)$. Also, every compact subset of $Y$ is metrizable. Then, by the proof of Theorem 1.1, we have

**Corollary 1.4.** The following are equivalent.

1. $Y$ is the closed image (resp. closed s-image) of a metric space, and each closed (or countable closed) metric subspace is locally compact.

2. $Y$ is the closed image (resp. closed s-image) of a locally compact, metric space.

Let $X$ be a space and let $\mathcal{C}$ be a covering (not necessarily closed or open) of $X$. Then $X$ has the weak topology with respect to $\mathcal{C}$ provided that, for $A \subseteq X$, if $A \cap C$ is closed in $C$ for all $C \in \mathcal{C}$, then $A$ is closed in $X$. If $\mathcal{C}$ is
a closed covering, as a stronger notion, let us recall that $X$ has the hereditarily weak topology with respect to $\mathcal{C}$, or equivalently $X$ is dominated by $\mathcal{C}$, provided that, for every $\mathcal{C}' \subset \mathcal{C}$, if $A \subset \bigcup \mathcal{C}'$ and $A \cap C$ is closed in $C$ for all $C \in \mathcal{C}'$, then $A$ is closed in $X$.

Recall that every space having the weak topology with respect to an increasing, countable closed cover is dominated by the cover. Also, every CW-complex is dominated by the cover of all finite subcomplexes.

Not every space dominated by a countable cover of compact metric subspaces can be decomposed into a $\sigma$-discrete subset and a locally compact metric subspace. Indeed, this can be seen by the countable CW-complex obtained from the topological sum of countably many triangles, $\Delta a_i b_i c_i$, by identifying all of segments, $\overline{a_i b_i}$, to a segment. So, as an application of Theorem 1.1, we shall consider the decomposition of spaces having the weak topology with respect to a closed covering of locally compact subspaces.

Recall that a space is Fréchet if, whenever $x \in \overline{A}$, then there is a sequence in $A$ converging to the point $x$.

**Proposition 1.5.** For a space $Y$ having a closed cover $\mathcal{J}$ of locally compact subspaces, we define the following properties.

(a) $Y$ is dominated by $\mathcal{J}$.

(b) $Y$ has the weak topology with respect to $\mathcal{J}$ such that $\mathcal{J}$ is point-countable, e.g., $Y$ is the quotient Lindelöf image of a locally compact paracompact space.
Then \( Y \) is the union of a discrete closed subspace and a locally compact subspace if \( Y \) satisfies (1) or (2) below.

(1) Either (a) or (b) holds, and \( Y \) is the closed image of a paracompact bi-k-space.

(2) \( Y \) is a Fréchet space, and (b) holds.

Proof. Case (1): We will prove that \( Y \) is the closed image of a locally compact paracompact space. Hence, by [10; Theorem 4], \( Y \) is decomposed as a discrete closed subspace and a locally compact subspace. To prove that, from Theorem 1.1, it suffices to show that every closed, \( \sigma \)-compact \( M \)-subspace \( F \) of \( Y \) is locally compact. If \( Y \) satisfies (a), then it is easy to check that every compact subset of \( Y \) is contained in a finite union of elements of \( J \). Then \( F \) is contained in a countable union of elements \( F_i \in J \).

Since \( F \) is closed, it is obvious that \( F \) has the weak topology with respect to a countable, closed cover \( \{ F \cap F_i; i = 1, 2, \cdots \} \) of locally compact subspaces. This shows that the case (a) reduces to (b). So we assume that \( Y \) satisfies (b). It follows from [14; Lemma 6], for each \( k \)-sequence \( (A_n) \) in \( Y \), that some \( A_{n_0} \) is contained in a finite union of elements of \( J \), so \( A_{n_0} \) is locally compact. Hence, in view of the proof of Theorem 1.1, \( Y \) is the closed image of a locally compact paracompact space.

Case (2): Let \( D = \{ y \in Y; y \notin \text{int}(\cup J') \} \) for any finite \( J' \subset J \). Then it is sufficient to show that \( D \) is a discrete closed subset of \( Y \). To see that \( D \) is discrete in \( Y \), suppose not. Then there exist a sequence \( y_n \in D \) and a point \( y_0 \notin D \) with \( y_n \to y_0 \). Let \( K = \{ y_n; n \in N \} \cup \{ y_0 \} \), and
\{F \in J; F \cap K \neq \emptyset \} = \{F_1, F_2, \ldots \}, \text{ and let } T_n = \bigcup_{i=1}^{n} F_i.

Since \( y_n \in D \) (hence, \( y_n \in \overline{Y - T_n} \)), there exist sequences \( S_n = \{y_{nj}; j = 1, 2, \ldots \} \) converging to \( y_n \), and \( S_n \cap T_n = \emptyset \).

Since \( y_0 \in \bigcup_{n=1}^{\infty} S_n \) and \( y_n \neq y_0 \), there exists a sequence \( S = \{y_k'; k = 1, 2, \ldots \} \) converging to \( y_0 \) with \( y_k' \in S_{n_k} \).

But, each convergent sequence together with the limit point (hence, a compact subset) of \( Y \) is contained in a finite union of elements of \( J \). Thus, for some \( n_0 \) and a subsequence \( S_0 = \{y_k(j); j = 1, 2, \ldots \}, S_0 \cup \{y_0\} \subset T_{n_0} \). Thus, for any \( n = m > n_0, S_m \cap T_m \neq \emptyset \), a contradiction. Hence \( D \) is discrete in \( Y \).

2. Fréchet Spaces and Lašnev Spaces

Not every Fréchet space having the weak topology with respect to a point-finite closed cover of metric (hence Lašnev) subspaces is Lašnev. Indeed, let \( X \) be the upper half plane. For each real number \( r \) and each \( n \in \mathbb{N} \), let \( \{(x,y); y = |x - r| < \frac{1}{n}\} \) be a basic neighborhood of \((r,0)\), and let the other points be isolated. Then \( X \) is a first countable (hence Fréchet) space having the weak topology with respect to a point-finite clopen cover of metric subspaces. But \( X \) is not Lašnev, for it is not normal. So, in terms of weak topology, we shall consider conditions that imply every Fréchet space with Lašnev pieces is Lašnev.

**Proposition 2.1.** Let \( X \) be a Fréchet space having the weak topology with respect to a closed cover \( J \) of Lašnev subspaces. If (1) or (2) below holds, then \( X \) is Lašnev.
(1) \( J \) is countable.

(2) \( J \) is point-countable and each element of \( J \) is a separable space.

Proof. Case (1): Let \( J = \{F_n; n \in N\}, C_n = \bigcup_{i=1}^{n} F_i \), and let \( M_n = C_n - C_{n-1}, C_0 = \emptyset. \) Let \( M \) be the topological sum of \( M_n (n \in N) \), and let \( f: M \to X \) be the obvious map.

Since \( X \) is Fréchet space having the weak topology with respect to \( \{C_n; n \in N\} \) with \( C_n \subseteq C_{n+1} \), by the proof of F. Siwiec [12; Proposition 2(a)], we show that \( f \) is a closed map without any topological property of \( F_n \). Since each \( F_n \) is now Lašnev, so is \( M \). Then \( X \) is Lašnev.

Case (2): Every separable Lašnev space is obviously the closed image of a separable metric space. Thus, each element of \( J \) is an \( N_o \)-space [8], that is, a space with a countable \( k \)-network. Thus \( X \) has the weak topology with respect to a point-countable cover of \( N_o \)-subspaces. Since \( X \) is Fréchet, by [5; Corollary 8.9], \( X \) is the topological sum of \( N_o \)-spaces. Hence, it is sufficient that every closed, \( N_o \)-subspace \( S \) of \( X \) is Lašnev. To see this, let \( N \) be a countable \( k \)-network for \( S \), that is, whenever \( K \subseteq U \) with \( K \) compact and \( U \) open in \( S \), then \( K \subseteq \bigcup N' \subseteq U \) for some finite \( N' \subseteq N \). We assume that each element of \( N \) is closed in \( S \), and that \( N \) is closed under finite unions and intersections. Now, let \( K \) be any compact subset of \( S \), and each \( N_n \in N \) contain the set \( K \), and let \( K_n = \bigcup_{i=1}^{n} N_i \) for each \( n \). Then \( K_n \in N \) and \( (K_n) \) is a \( k \)-sequence in \( S \) with \( K = \bigcap_{n=i}^{\infty} N_n \). On the other hand, the closed subset \( S \) of \( X \) has the weak topology with respect to a point-countable closed cover.
\[ C = \{ S \cap F ; F \in J \} \]. Hence by [14; Lemma 6], some \( K \) is contained in a finite union of elements of \( C \). Since each element of \( C \) is Lašnev, so is \( K \). This implies that 
\[ N' = \{ N \in N; N \text{ is Lašnev} \} \] is a k-network for \( S \). Thus, \( S \) has the weak topology with respect to the countable closed cover \( N' \), because \( S \) is Fréchet (hence a k-space) and each compact subset of \( S \) is contained in an element of \( N' \). Hence, by (1), \( S \) is a Lašnev space. That completes the proof.

F. Siwiec [12] showed that closed images of locally compact, separable metric spaces are precisely the hemicompact Fréchet spaces in which every compact subset is metrizable. As for closed s-images of locally compact metric spaces, we have the following characterization and a relationship between closed s-images and pseudo-open s-images.

Recall that a map \( f: X \to Y \) is pseudo-open [1] (i.e., hereditarily quotient) if for any \( y \in Y \) and for any neighborhood \( U \) of \( f^{-1}(y) \), \( y \in \text{int} \ f(U) \).

**Theorem 2.2.** The following are equivalent.

1. \( Y \) is the pseudo-open s-image of a metric space, and each closed (or countable closed) metric subspace is locally compact.
2. \( Y \) is the pseudo-open s-image of a locally compact, metric space.
3. \( Y \) is the closed s-image of a locally compact, metric space.
4. \( Y \) is a Fréchet space having the weak topology with respect to a point-countable cover of compact metric subspaces.
Proof. Since every closed map is pseudo-open, (3) \implies (2) is obvious.

(2) \implies (1): Let \( f: X \rightarrow Y \) be a quotient s-map with \( X \) a locally compact, metric space. Let \( F \) be a closed, metric subspace of \( Y \). Then \( g = f|f^{-1}(F) \) is a quotient s-map from a locally compact space \( f^{-1}(F) \) onto a metric space \( F \). Thus, \( F \) is locally compact by [9; Propositions 3.3(d) and 3.4(a)]. Thus every closed metric subspace of \( Y \) is locally compact.

(4) \implies (3): From Proposition 2.1, \( Y \) is Lašnev. Thus, as in the proof of Proposition 1.5, \( Y \) is the closed image of a locally compact, metric space (under \( f \)). Since \( Y \) has the weak topology with respect to a point-countable cover of compact subsets, every \( \exists f^{-1}(y) \) is Lindelöf by [13; Remark 4]. Thus \( Y \) is the closed s-image of a locally compact, metric space.

(1) \implies (4): Let \( f: X \rightarrow Y \) be a pseudo-open s-map with \( X \) metric. Since each closed metric subspace of \( Y \) is locally compact, as in the proof of Theorem 1.1 or Corollary 1.4, \( X \) has a locally finite closed cover \( J \) each of whose element is contained in the inverse-image of a compact subset of \( Y \). Since \( J \) is a locally finite closed cover of \( X \) and \( f \) is a quotient s-map, \( Y \) has the weak topology with respect to a point-countable cover \( f(J) \) each of whose closure is compact, hence separable metric by [3; Corollary 3]. While, \( Y \) is Fréchet, for every pseudo-open image of a metric space is Fréchet [1]. Thus, by [5; Corollary 8.9], \( Y \) is the topological sum of \( \mathcal{N}_0 \)-spaces. But \( Y \) has the weak topology with respect to a point-countable cover each of whose closure is
compact. Hence, by the same way as in the proof of Proposition 2.1, \( Y \) has the weak topology with respect to a point-countable cover of compact metric subspaces. That completes the proof.

Since every quotient map onto a Fréchet space is pseudo-open [1], using a decomposition theorem [10; Theorem 4] of closed images of locally compact spaces, we have

**Corollary 2.3.** Every Fréchet space which is the quotient \( s \)-image of a locally compact metric space is a Lašnev space which can be decomposed into a discrete closed subspace and a locally compact metric subspace.

We remark that not every Fréchet space which is the quotient \( s \)-image of a metric space is either Lašnev or is the union of a discrete closed subspace and a metric subspace; see [5; Example 9.4], which is Lindelöf, non-separable, and has a point-countable base (hence, it is the open \( s \)-image of a metric space by [11]). As for decompositions of Lašnev spaces, N. Lašnev [6] showed that not every countable Lašnev (hence, \( N_\sigma \)) space is the union of a discrete closed subspace and a metric subspace, but every Lašnev space is at least decomposed as a \( \sigma \)-discrete subset and a metric subspace.

Thus, in view of Corollary 2.3 and Proposition 1.5 (case (2)), we pose the following problem concerning Fréchet spaces.

**Problem 2.4.** Is every Fréchet space which is the quotient image of a separable metric space (i.e., Fréchet
$\mathcal{N}_0$-space) Lašnev, or at least decomposed as a $\sigma$-discrete subset and a metric subspace? How about every Fréchet space dominated by a cover of compact metric subspaces (e.g., Fréchet CW-complex)?

If the former is affirmative, then every Fréchet space which is the quotient s-image of a locally separable metric space is Lašnev, or at least decomposed as a $\sigma$-discrete subset and a metric subspace, because such spaces are characterized as the Fréchet spaces which are the topological sum of $\mathcal{N}_0$-spaces by [5; Proposition 8.8] and [8; Corollary 11.5].

As for the latter, if every Fréchet space dominated by a cover of compact metric subspaces is Lašnev, then such a space can be decomposed into a discrete closed subspace and a locally compact metric subspace by Proposition 1.5 (case (1)).

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