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ON Z AND Z*-SPACES

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1. Introduction

The condition Z was introduced by Zenor in [15] and studied by Mack in [11], and the condition Z* was introduced by the author in [3]. The primary purpose of this paper will be to study the C- and C*-embedding properties of Z and Z* spaces and in particular to prove the following theorem.

Theorem. A Tychonoff Z*-space* (Z*-*space) is* C*-*embedded in every Tychonoff* Z*-space* (Z*-*space) it is embedded in as a closed set iff it is almost Lindelöf (pseudocompact).*

2. Definitions and Notation

Definition 1. A topological space X is a Z-space (Z*-space) if for $F, Z \subset X$, $F \cap Z = \emptyset$, F closed, and Z a zero set, then F and Z are completely separated in the terminology of [8] (every closed set is a generalized Z_σ; i.e. for F closed G open, and $F \subset G$, there exists H, a countable union of zero sets, such that $F \subset H \subset G$).

Example 1. The Tychonoff plank satisfies Z* but not Z. On the other hand $\Omega = W^* \times W^* - (w_1, w_1)$ satisfies Z but not Z* where W* is the set of ordinals up to and including the first uncountable ordinal [15].

Definition 2. Two disjoint sets $A, B \subset X$ are functionally separated if there exists a continuous function to the real line such that $f(A) \cap f(B) = \phi$. If for $x, y \in X$, $x \neq y$,

[x] and [y] are functionally separated then X is said to be functionally Hausdorff [3]. If for $x \in X$, $F \subset X$, $x \notin F$, and F closed (K compact in weak topology $K \subset X$, $F \subset X$, $K \cap F = \emptyset$, and F closed) [x] and F (K and F) are functionally separated then X is said to be functionally regular (k-functionally regular).

In general, the notation of Gillman and Jerison [8] will be used. However $(X,\mathcal{W})$ will be used to represent the weak topology for $(X,\mathcal{J})$ and z-embedding is defined in [5].

3. Basic and Known Properties of Z- and Z*-Spaces

In the theorem below we will list basic and known properties of Z- and Z*-spaces. Many of the proofs will be omitted.

Theorem 1. *The following relations hold.*

(a) *A space is normal iff it satisfies* Z *and* Z*.

(b) *A countably paracompact* Z*-*space is normal.*

(c) *A functionally regular* Z-*space is completely regular. A functionally Hausdorff* Z*-*space is* k-*functionally regular.*

(d) *A pseudocompact* Z-*space is countably compact (Proof of Tychonoff case in* [15]*). A countably compact space* [15] *or a* P-*space is a* Z-*space.*

(e) *A* σ-*compact functionally Hausdorff space satisfies* Z*.

(f) *Closed sets of* Z*-*spaces are* Z*-*spaces and* z-*embedded closed sets of* Z-*spaces are* Z-*spaces.*

(g) *Disjoint closed* C-*embedded subsets of* Z*-*spaces are completely separated and hence finite unions of disjoint closed* C-*embedded sets are* C-*embedded.*

(h) *Closed countable unions of* z-*embedded zero sets in* Z-*spaces are* z-*embedded and hence* C-*embedded.*

(i) *Both* Z *and* Z* *are preserved by* GZ-*mappings (mappings that take zero sets onto generalized zero sets).*

(j) *Every* WZ-*mapping from a* Z-*space is a* Z-*mapping* [15].

Proofs. (a) If A and B are disjoint closed sets of X there exists a Z_σ set H such that $A \subset H \subset B$. By the Z property there exists a zero set containing B and disjoint from H. A second application of the Z property establishes that A and B are completely separated so X is normal. The converse is immediate given that a space is normal iff every closed set is a generalized zero set (folklore). (b) In the proof of part (a), first sentence, let $H = \cup Z_n$. We use the countable paracompactness of X to establish $Z_n \subset G_n \subset \overline{G}_n \subset \sim B$ where G_n is open. An application of countable paracompactness to A shows that A and B are contained in disjoint open sets. (e) From [2] Lindelöf functionally Hausdorff spaces are k-functionally regular. Let A and B be disjoint closed sets. Since A is σ-compact there is a Z_σ containing A and disjoint from B. (g) Let A and B be disjoint C-embedded closed sets of a Z*-space. If H is a Z_σ set containing A and disjoint from B there is a zero set containing B disjoint from H by the C-embedding of B. The C-embedding of A establishes that A and B are completely separated. For any finite family $\{F_n\}$ of disjoint C-embedded closed sets there is a family of disjoint zero sets $\{Z_n\}$ such that each $F_n \subset Z_n$. The C-embedding of $\cup F_n$ will follow. (h) Let $F = \cup Z_n$ where each Z_n is a z-embedded zero set and hence a C-embedded zero set [5]. Let A and B be disjoint zero sets of F and let $A_n = A \cap Z_n$. By the z-embedding of Z_n, A_n is a zero set of X and hence is completely separated from B by the Z-property, and there is a zero set containing B disjoint from $A = \cup A_n$; again we apply the Z-property so that A and B are completely separated.

Remarks on Theorem 1. (g) The space Ω has two disjoint closed C-embedded subsets that are not completely separated [8] so the result does not apply to Z-spaces. Any attempt to replace finite by countable runs into the difficulty of the Tychonoff plank which has a countable closed discrete set which is not even C*-embedded. This shows also that unlike Z-spaces z-embedded closed sets are not necessarily C- or C*-embedded. (j) Zenor [15] showed that Z-mappings (mappings such that the image of zero sets are closed sets) are identical with closed maps in normal spaces. Since every GZ-mapping is a Z-mapping and closed sets of normal spaces are generalized zero sets (folklore) GZ-mappings are identical with closed maps in normal spaces.

4. The Transition from Complete Regularity to Normality

Hewitt [10] has shown that a Tychonoff space is C*-embedded in every Tychonoff space it is embedded in iff it is alsmot compact; the theorem below gives a comparison theorem for complete regularity and normality. In the above result of Hewitt and the theorem below, we may replace C*-embedded by C-embedded.

Theorem 2. A Tychonoff (locally compact) space X *is* C^**-embedded in every Tychonoff (locally compact) space* X *is embedded in as a closed set iff* X *is almost compact. For a* T_4*-space every closed embedding is a* C*-embedding.*

Proof. It is evident that the result on Tychonoff spaces follows from Hewitt's result. If a Tychonoff space X is not almost compact there is a compactification KX in which X is not C^*-embedded. By problem 9K of [8] there exists Y such that $KX = \beta Y \sim Y$. Then X is not C^*-embedded in $X \cup Y$ but is closed in $X \cup Y$. The proof for the locally compact case is almost the same. The last result is a classical result of Urysohn.

There is clearly a large gap between complete regularity and normality in regard to C and C^*-embedding of closed sets. With Theorems 3 and 4 we will help fill this gap by showing that every closed embedding of a $Z(Z^*)$-space in a Tychonoff $Z(Z^*)$-space is a C- or C^*-embedding iff the space is almost-Lindelöf (pseudocompact). The method of proof for these two conditions will be different and also distinct from the argument in Theorem 2. However, one could prove the special case involving property Z when the space is countably compact using the argument of Theorem 2. The method we use is closely related to but more involved than the one used for investigating P-spaces [1] of Gillman and Jerison.

We note for z-embeddings the condition on X for Tychonoff spaces in Theorem 2 is almost compact or Lindelöf. This is a consequence of a result of Blair and Hager [5].

Remark. In the next theorem we do not have to specify that the space X is a Z-space except to insure that there is a Tychonoff Z-space in which it is embedded as a closed set. For instance no almost compact space that is not countably compact can be embedded as a closed set of a Z-space. So such a space would vacuously have the property that it is C-embedded in every Tychonoff Z-space it is embedded in as a closed set. It would be interesting to characterize the closed subspaces of Z-spaces. In the statement of the next theorem, we may replace C*-embedded by z-embedded or C-embedded.

Theorem 3. *A Tychonoff* Z *space is* C*-*embedded in every Tychonoff* Z *space that it is embedded in as a closed set iff it is almost Lindelöf. (Note: A space is almost Lindelöf if given two disjoint zero sets one is Lindelöf.)*

We will need several lemmas.

Lemma 3A. *A space* X *is almost Lindelöf iff* $\beta X \sim X$ *does not contain two disjoint closed sets of* βX *neither of which is contained in a zero set of* βX *that is contained in* $\beta X \sim X$.

The above lemma was proved in [1].

Lemma 3B. *A one point Tychonoff extension of a* Z-*space* X *is a* Z-*space.*

Proof. The Tychonoff space $X \cup [y] = Y$ can be considered as the continuous image of $X \cup K$, where K is compact and $X \cup K$ is a subset of βX and the continuous mapping f is the extension of a homeomorphism of X onto itself. See [8].

Let $A, Z \subset Y$, $A \cap Z = \emptyset$, A closed in Y and Z a zero set of Y. If $y \in Z$ then $K \subset k^{-1}(Z)$ and there exists a zero set H of $X \cup K$ containing $f^{-1}(A) = A$ and disjoint from K. Furthermore there exists a zero set Q of $X \cup K$ disjoint from $f^{-1}(Z) \cap X$ and containing A so that $Q \cap H$ is a zero set disjoint from $f^{-1}(Z)$ containing A. Then $f(H \cap Q) = H \cap Q$ is a zero set of Y disjoint from Z and containing A. Suppose $y \notin Z$. There exists a zero set H of X such that $H \cap Z = \emptyset$ and $A \cap X \subset H$. By the C*-embedding of X in $X \cup K$ these zero sets can be extended to $X \cup K$ and furthermore in such a way that the zero set extension Q of H contains K. Then $f(Q) = (Q \cap X) \cup [y]$ is a zero set of Y containing A and disjoint from Z.

Lemma 3C. *Let* $(X,\mathcal{J})$ *be a Tychonoff* Z*-space and* $(Y,\mathcal{S})$ *a one point Tychonoff extension of* $(X,\mathcal{J})$ *where* $Y = X \cup [y]$. *The space* $(Y,\mathcal{U})$ *with subbase all open sets of* $(Y,\mathcal{S})$ *and all complements of zero sets of* $(X,\mathcal{J})$ *which are disjoint from zero sets of* $(Y,\mathcal{S})$ *containing* $[y]$ *is a Tychonoff* Z*-space.*

Proof. Let H be a zero set of $(X,\mathcal{J})$ disjoint from Z, a zero set of $(Y,\mathcal{S})$ such that $y \in Z$. We will show that H is a zero set of $(Y,\mathcal{U})$. There is a continuous function f on X such that $f^{-1}(1) = H$ and $f^{-1}(0) = Z \cap X$. We define f^* so that $f^*(x) = f(x)$ for $x \in X$ and $f^*(y) = 0$. If A is closed in the reals and $0 \in A$ then $f^{-1}(A)$ is closed in X; $y \in f^{*-1}(A)$ so that $f^{*-1}(A)$ is closed in Y. If $0 \notin B$, B closed in the reals, $f^{-1}(B)$ is a zero set in X, since Z is disjoint from $f^{-1}(B)$. So $f^{-1}(B)$ is closed in $(Y,\mathcal{U})$. Thus f^* is continuous and H is a zero set of $(Y,\mathcal{U})$. To show the complete regularity of $(Y,\mathcal{U})$ it will suffice to show that if $y \notin F$ and F is closed in $(Y,\mathcal{U})$ but not closed in $(Y,\mathcal{S})$ then F and $[y]$ are completely separated in $(Y,\mathcal{U})$. By construction of $(Y,\mathcal{U})$ there is a zero set H of $(Y,\mathcal{U})$ such that $F \subset H$ and $y \notin H$. So F and y are completely separated in $(Y,\mathcal{U})$ and $(Y,\mathcal{U})$ satisfies Z by Lemma 3B.

Lemma 3D. *Let* $(Y,\mathcal{S})$ *and* $(Y,\mathcal{U})$ *be constructed as in Lemma* 3C *and let* Z *be a zero set of* $(Y,\mathcal{U})$ *such that* $y \in Z$. *Then there is a zero set* M, *of* $(Y,\mathcal{S})$ *such that* $y \in M \subset Z$. *If* $[y]$ *is a zero set of* $(Y,\mathcal{U})$ *then* $[y]$ *is a zero set of* $(Y,\mathcal{S})$.

Let Z be a zero set of $(Y,\mathcal{U})$ containing $[y]$. We wish to show that there is a zero set M of $(Y,\mathcal{S})$ such that $y \in M \subset Z$. In $(Y,\mathcal{U})$, $\sim Z = \cup F_i$ where each F_i is a zero set and $F_i \subset F_{i+1}$. If in $(Y,\mathcal{S})$ there exists F_i such that if $j \geq i$, F_j is not closed, then there exists a zero set M of $(Y,\mathcal{S})$ such that $y \in M \subset Z$. This follows from the construction of $(Y,\mathcal{U})$. If each F_i is closed in $(Y,\mathcal{S})$, then by the complete regularity of $(Y,\mathcal{S})$ there exists a zero set M such that $y \in M \subset Z$. It is clear that since $[y]$ is not a zero set of $(Y,\mathcal{S})$, then $[y]$ is not a zero set of $(Y,\mathcal{U})$.

Definition 4. A set X, $X \subset Y$, satisfies Z in regard to a space Y if for F closed in X and Z a zero set of Y such that $F \cap Z = 0$ there is a zero set H of Y such that $F \subset H \subset \sim Z$.

Example 2. If X is an uncountable discrete space and Y a one point extension by a point y such that the neighborhoods of y are complements of countable sets (the one point compactification of X) then X satisfies (does not satisfy) Z in regard to Y.

Lemma 3E. *Let* $(X,\mathcal{J})$ *be Tychonoff,* Z, *and not almost-Lindelöf. Then there is a one point Tychonoff extension* Y *of* X *which satisfies* Z *and is such that* $(X,\mathcal{J})$ *satisfies* Z *with respect to* Y *and* X *is not* C*-*embedded in* Y.

Proof. Let A and B be the disjoint closed sets of Lemma 3A and consider the quotient map of $X \cup A \cup B$ with each $x \in X$ an equivalence class and $A \cup B$ an equivalence class. Let $f(x) = x$ and $f(A \cup B) = y$ under the quotient map f and designate this topology as $(Y,\mathcal{S})$. Then as in Lemmas 3C and 3D construct the topology $(Y,\mathcal{U})$. We have shown in these two Lemmas that $(Y,\mathcal{U})$ is a one point Tychonoff extension of $(X,\mathcal{J})$ that satisfies Z. If F is closed in $(X,\mathcal{J})$ and disjoint from a zero set Z containing Y, then since $(Y,\mathcal{U})$ satisfies Z, and F is closed in $(Y,\mathcal{U})$, X satisfies Z in regard to Y. Since neither A nor B nor $A \cup B$ is a zero set, X is not C*-embedded in Y. Suppose X were C*-embedded in the modified quotient space of $X \cup A \cup B$ of Lemma 3A where each $x \in X$, A and B are the equivalence classes forming a space $X \cup [a] \cup [b]$, and where for the quotient map f, $f(x) = x$, $f(A) = a$ and $f(B) = b$. We modify this quotient space analogous to the process in Lemmas 3C and 3D by closing zero sets of $(X,\mathcal{J})$ disjoint from zero sets of $X \cup [a] \cup [b]$ containing $[a] \cup [b]$. Even if X were C*-embedded in the modified $X \cup [a] \cup [b]$, X would not be C*-embedded in $Y = (Y,\mathcal{U})$ which we could obtain by identifying a and b.

Lemma 3F. *Let* Y *satisfy* Z *and let* $Y = X \cup [y]$ *with* X *satisfying* Z *with respect to* Y. *Then* $Q = (Y \times D^*) - (y,d^*)$ *satisfies* Z *where* $|D| > |Y|$ *and* $D^* = D \cup [d^*]$ *where each* $d \in D$ *is isolated in* D^* *and a set containing* d^* *is open iff the cardinality of its complement is less than or equal to* $|Y|$.

Proof. Let A be closed and let Z be a zero set of Q such that $A \cap Z = \emptyset$. For Z a zero set of Q such that $(x,d^*) \in Z$, all but $|Y|$ of $(x,d) \in Z$ for $d \in D$. If $(x,d^*) \in Z$ then there is at most $|Y|$ members of D such that $(x,d) \notin Z$. So there exists a subset D_1, of D^*, $d^* \in D_1$, and $|D_1| = |D|$ and $((Z \cap X) \times D_1) \subset Z$. Furthermore we can choose D_1 so that if $(y,d) \in Z$ for some $d \in D_1$ then $(y,d) \in Z$ for all $d \in D_1$. This follows from Lemma 3D where it is shown that $[y]$ is not a zero set of Y so that for $H \subset X$, H and $H \cup [y]$ can not both be zero sets of Y. So that either (1) $(Z \cap X) \times D_1 = Z \cap (Y \times D_1)$ or (2) $((Z \cap X) \times D_1) \cup (D_1 \cap Q) = Z \cap (Y \times D_1)$. For a closed set A such that $(x,d^*) \notin A$ there are at most $|Y|$ members of D such that $(x,d) \in A$. So we may choose D_1 in addition so that if $(x,d^*) \notin A$, then $(x,d) \notin A$ for $d \in D_1$. In case (1) above $A \cap (Y \times D_1) \subset ((A \cap X) \times D_1) \cup (D_1 \cap Q)$ a closed set. In case (2) above $A \cap (Y \times D_1) \subset ((A \cap X) \times D_1)$ a closed set. In either case there is a zero set Z_2 disjoint from the zero set $Z_1 = Z \cap (Y \times D_1)$ and containing the closed set $A \cap (Y \times D_1)$ by Lemma 3E. In general there will be points of A and Z outside of $Y \times D_1$. Since d is clopen in D for $d \notin D_1$, $Z_d = Z \cap (Y \times d)$ is a zero set of Q and there is a zero set $H_d \supset A \cap (Y \times d)$ disjoint from $Z \cap (Y \times d)$ by Lemma 3E. Then $Z = Z_1 \cup\cup \{Z_d: d \in D \sim D_1\}$ is disjoint from $Z_3 = Z_2 \cup\cup \{H_d: d \in D \sim D_1\}$ which is a zero set containing A by Problem 1A [8].

Lemma 3G. *In a* Z-*space* z-, C*- *and* C-*embeddings are equivalent for closed sets.*

Proof. Blair and Hager showed that a z-embedded set is C-embedded iff it is completely separated from any disjoint zero set.

Proof of theorem. If X is not almost-Lindelöf, we construct the one point extension of Lemma 3E and then the deleted product space of Lemma 3F which satisfies Z. The set X is not C*-embedded in Q since there exists f continuous and bounded on X which can not be extended to Y. Any extension of this function must be copied on at least one $X \times d$, in fact, for all but $|Y|$ of these d since zero sets containing d* must contain all but $|Y|$ points of D*. A copy of f on $X \times d$ can not be extended to $Y \times d$. So f cannot be extended to Q. We then note that in a Z-space for closed sets z-, C*-, C-embedding are equivalent by Lemma 3G. We conclude the proof by noting that in [3] it was proved that every closed almost Lindelöf subset of a Tychonoff Z-space is C-embedded.

Example 3. We note that the examples in Example 1 being almost compact are almost Lindelöf.

Theorem 4. *Every closed embedding of a Tychonoff* Z^**-space* X *in a Tychonoff* Z^**-space is a* C^**-embedding iff* X *is pseudocompact.*

Proof. In [3] it was shown that closed pseudocompact subsets of Z^*-spaces are C-embedded. Let X be Tychonoff Z^* and not be pseudocompact. Using the notation of Gillman and Jerison, let T be the Tychonoff plank, where $T = N^* \times W^* \sim (n^*, w_1)$. Let $Y = X \cup T$ where $X \cap T = N$ where N is both the countable closed discrete space of the plank and a C-embedded copy of N in X which exists since X is not pseudocompact [8]. A set will be a member of the subbase for Y if it is the complement of a closed set in X or T. We will show that X is not C^*-embedded in $Y = X \cup T$ and Y satisfies Z^* and is Tychonoff. If f is bounded and continuous on N but does not have an extension to all of T, then any extension of f to X and there is at least one, in fact, a bounded one, can not be extended to Y. So X is not C^*-embedded in $X \cup T$. In order to show that Y satisfies Z^* we first show that X is z-embedded in Y. Let Z be a zero set of X and let $x_1, x_2, \cdots x_n \cdots$ be points of N not contained in Z. There exists a zero set Z_n containing all points of the above sequence except the first $n-1$ and Z_n is disjoint from Z. This is possible since N is C-embedded in X. For this reason it is no restriction to have $\cap_{n=1}^{\infty} Z_n = \emptyset$. The zero set $Z \cup Z_n$ of X can be extended to a zero set Q_n of Y such that $Q_n \cap X = Z \cup Z_n$ since they contain all but a finite number of points of N. Since $Z = \cap_{n=1}^{\infty}(Z \cup Z_n)$, Z can also be extended to a zero set Q of Y such that $Q \cap X = Z$.

Hence, X is z-embedded in Y. At this point it is clear that Y is a Tychonoff space. It remains to show that Y satisfies Z*. Let A and B be disjoint closed sets of Y. If H is a Z_σ of T such that $A \cap T \subset H \subset T \sim (B \cap T)$ then there is a Z_σ set Q of Y such that $A \cap T \subset Q \subset T \sim (A \cap T)$. Similarly we can find a Z_σ, R of Y such that $A \cap X \subset R \subset X \sim (B \cap X)$. The set $A \cap T$ is the countable union of compact sets with a countably compact set. Each of these compact sets is contained in a zero set disjoint from $B \cap X$. Furthermore W, a zero set in T, is a zero set in $Y = T \cup X$. For the function $1/f$ has a positive extension from N to all of X where f is the function of T such that $f(w_1,n) = 1/n$ and $f(w_1,n^*) = 0$, so that f has an extension to X such that for $x \in X$, $f(x) \neq 0$. Thus there is a Z_σ set, S containing $A \cap T$ and disjoint from $B \cap X$. Since by similar reasoning there is a Z_σ containing $B \cap T$ and disjoint from $A \cap X$, there is a zero set disjoint from the countable union of compact sets of $B \cap T$, and its intersection with $Y \sim W$ gives a Z_σ set J containing $A \cap X$ but disjoint from $B \cap T$. Then $(Q \cap S) \cup (R \cap J)$ is a Z_σ containing A and disjoint from B.

By a modification of the above proof we can obtain the following more general result.

Theorem 4A. *A* T_1, Z*-*space* X *is* C*-*embedded in every* T_1, Z*-*space it is embedded in iff* X *is pseudocompact.*

In this theorem and Theorem 4, we may replace C*-embedding by C-embedding.

5. On Z- and Z*-Closed Spaces

One of the questions investigated by the minimal topologists is when can a space having property Q be embedded as a closed nowhere dense set of a Q-closed space [4].

Definition 5. Let Q be a property. Then P = NF[Q] means that a Q space can be C*-embedded as a nowhere dense closed set of a Q-closed space iff it satisfies property P.

The following table gives a comparison of several properties in the context of a T_1-space.

	P = NF[Q]	Q
1.	Compact	Completely regular
2.	Closed sets, Lindelöf in weak topology	k-functionally regular (see remark below)
3.	Weak topology realcompact	Functionally regular
4.	No restriction	Functionally Hausdorff
5.	Pseudocompact = countable compact	Functionally Hausdorff and Z
6.	Weak topology Lindelöf = closed sets Lindelöf in weak topology	Z*

Item 1 is due to the fact that completely regular closed spaces are compact and is only put in as a basis of comparison. Items 2, 3, and 4 were proved in [2], and items 5 and 6 will be proved below. In the case of item 2 the embedded space and the space embedded in while being k-functionally regular the space embedded in is FH-closed rather than k-functionally regular closed. For the other conditions Q-closed is equivalent to FH-closed. Since in Z-spaces every continuous function may be constant the context for Z-spaces in this section will be functionally Hausdorff. First we will prove that Z-closed and Z*-closed are identical with (FH-closed) functionally Hausdorff closed = weak topology compact for Z- and Z*-spaces.

Theorem 5. *A Z-space (Z*-space) is* FH-*closed iff it is* Z-*closed (Z*-closed).*

Proof. If a Z-space (Z*-space) is closed in every FH-space it is embedded then it is closed in every Z-closed (Z*-closed) space it is embedded in. Conversely, suppose $(X,\mathcal{J})$ is Z-closed (Z*-closed) and not FH-closed. Then we can construct a one point FH extension $Y = X \cup [y]$ such that X is dense and C*-embedded in Y [13]. We will show that Y satisfies Z(Z*) contradicting that X is Z-closed (Z*-closed). Let $(Y,\mathcal{V})$ be a one point Tychonoff extension of $(X,\mathcal{W})$ such that X is C*-embedded in Y. Let $(Y,\mathcal{U})$ be the topology with subbase consisting of the open sets of $(Y,\mathcal{V})$ and $(X,\mathcal{J})$. If $(X,\mathcal{J})$ satisfies Z, clearly $(X,\mathcal{W})$ satisfies Z, and so does $(Y,\mathcal{V})$. For if A is closed in Y and Z is a zero set of Y such that $A \cap Z = \emptyset$, then there exists a zero set H of X such that $A \cap X \subset H \subset X \sim Z$. If $y \in Z$, H will be a zero set of Y and if $y \notin Z$ there is a zero set of Y containing H and disjoint from Z by the C*-embedding of X in Y and by the fact that Y will be completely separated from Z in the latter case. If $Z = [y]$ we use the fact that the subbase used in the construction is a base and $A = B \cap F$ where B is the complement of a $\mathcal{J}$-open set and $y \notin F$ and F is closed in $(Y,\mathcal{V})$ and thus is completely separated from $[y]$ so that $(Y,\mathcal{U})$ satisfies Z.

We make the same construction for a Z*-space and show that the one point extension satisfies Z*. Let A and B be closed in $(Y,\mathcal{U})$ with $A \cap B = \emptyset$. There is a Z_σ of $(X,\mathcal{J})$, H, such that $A \cap X \subset H \subset X \sim B$. Then H or $H \cup [y]$ is a Z_σ of Y. The result follows if $y \notin B$ for then we can find a zero set containing y disjoint from B. (Z*-spaces are functionally regular). If $y \in B$ then we can find a zero set containing y and disjoint from A so that there is a Z_σ Q containing A but not $[y]$ so that $H \cap Q$ is a Z_σ such that $A \subset H \cap Q \subset\sim B$.

Theorem 6. *For* Q *property* Z *and functionally Hausdorff* P = NF[Q] *where* P *is pseudocompact or equivalently countably compact.*

Proof. If $(X,\mathcal{J})$ satisfies Z then so does $(X,\mathcal{W})$. Furthermore we note that we can extend Zenor's [15] result that a Tychonoff pseudocompact Z-space is countably compact for FH-spaces. Set $(X,\mathcal{W}) = \beta Y \sim Y$ and then construct the topology $(Z,\mathcal{U})$ on βY with subbase consisting of complements of closed sets of $(X,\mathcal{J})$ and open sets of βY. If X is countably compact so is $(Z,\mathcal{U})$ since Y under the construction of 9K of [8] is always countably compact so that $(Z,\mathcal{U})$ satisfies Z with weak topology compact and hence Z-closed. See Bourbaki [6, p. 138). From the construction $(X,\mathcal{J})$ is a nowhere dense closed set of $(X,\mathcal{U})$ and we can construct $(Y,\mathcal{U})$ so that X is C*-embedded and in fact C-embedded. Conversely, if $(X,\mathcal{J})$ is embedded as a nowhere dense closed C*-embedded set of a Z-closed space Y it is also C-embedded and as the weak topology for Y is compact, all functions on X are bounded so that $(X,\mathcal{J})$ is pseudocompact and countably compact.

Theorem 7. For Q *the property* T_1 *and* Z^*, $P = NF[Q]$ *where* P *is the property that the weak topology is Lindelöf.*

Proof. From [2] a k-functionally regular space $(X,\mathcal{J})$ can be C*-embedded as a nowhere dense closed set of an FH-closed space iff $(X,\mathcal{J})$ has the property that every closed set is Lindelöf in $(X,\mathcal{W})$. Since Z*-spaces are k-functionally regular spaces and Z*-closed spaces are FH-closed and in Z*-spaces closed sets of $(X,\mathcal{J})$ being Lindelöf in $(X,\mathcal{W})$ is equivalent to $(X,\mathcal{W})$ being Lindelöf, a Z*-space $(X,\mathcal{J})$ that is C*-embedded as a nowhere dense closed space of a Z*-closed space has the property that $(X,\mathcal{W})$ is Lindelöf. Conversely, if $(X,\mathcal{W})$ is Lindelöf we use this construction of $(Y,\mathcal{U})$ in Theorem 6 to obtain a Z*-space where $(X,\mathcal{J})$ is embedded as a nowhere dense closed C*-embedded subset. We only need to show that $(Y,\mathcal{U})$ satisfies Z*. Let A and B be closed in $(Y,\mathcal{U})$, $A \cap B = \emptyset$. Then $A = A_1 \cup A_2$, and $B = B_1 \cup B_2$ where A_1 and B_1 are closed in $(X,\mathcal{J})$ and A_2 and B_2 are closed in $(Y,\mathcal{V})$ the weak topology for $(Y,\mathcal{U})$. By the normality of $(Y,\mathcal{V})$ there is a zero set Z_2 of $(Y,\mathcal{V})$ containing A_2 and disjoint from B_2 and by the k-functional regularity of $(Y,\mathcal{U})$ [2] there is a zero set Z_1 containing A_2 and disjoint from B_1. By similar reasoning there is a zero set containing B_2 and disjoint from A_1 and hence a Z_σ, H_2 containing A_1 and disjoint from B_2. There exists a Z_σ in $(X,\mathcal{J})$ containing A_1 and disjoint from A_2 which can be extended to a Z_σ in $(Y,\mathcal{U}), H$ by the C*-embedding of X in Y such that $H_1 \cap A_2 = 0$. The set $H = (H_1 \cap H_2) \cup (Z_1 \cap Z_2)$ is a Z_σ such that $A \subset H \subset\!\sim B$.

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