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by

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# CAPACITY SPACES

**Harold R. Bennett, Wayne Lewis and Mladen Luksic**

In 1976 E. V. Scepin [$S_1$] defined a capacity on a $T_3$-space X to be a function $\varepsilon$ (called the capacity) from $X \times \mathcal{F}$ (where $\mathcal{F}$ is the class of all closed subsets of X) into the set of non-negative real numbers such that

(c-1) $\varepsilon(x,F) > 0$ if and only if $x \in Int(F)$ (Interior of F)

(c-2) if $F_1 \subseteq F_2$, then $\varepsilon(x,F_1) \leq \varepsilon(x,F_2)$

(c-3) if F is fixed, then $\varepsilon$ is continuous in the first variable, and

(c-4) if $\{F_\alpha\}$ is a family of closed sets linearly ordered by set inclusion, then $\varepsilon(x,\cap F_\alpha) = \inf_\alpha\{\varepsilon(x,F_\alpha)\}$, where $x \in X$ and each $F_\alpha \in \mathcal{F}$.

A space with a capacity is called a capacity space. If (X,d) is a metric space then a capacity may be defined on X by

$$\varepsilon(x,F) = d(x,X-F)$$

Thus a capacity space is a generalization of a metric space.

Recall that a closed subset A of space X is regularly closed if A = Cl(Int(A)) and the complement of a regularly closed set is called a regularly open set (where Cl(A) is the closure of A in X). Also, recall that a space is perfectly $\kappa$-normal if any two non-intersecting regularly closed sets have non-intersecting neighborhoods and every regularly closed set is the countable intersection of regularly open sets. Capacity spaces were evidently introduced as a tool to study perfectly $\kappa$-normal spaces.

Using (c-1) and (c-3) it can be shown that each capacity space is perfectly $\kappa$-normal.

It can be shown that Heath's sticker space, Example 1 of [H], is not perfectly $\kappa$-normal (and hence not a capacity space). It can also be shown that the Moore plane [W, exercise 4B] does have a capacity. Thus the property of having a capacity is a differentiating feature for the class of Moore spaces.

In [$S_1$] there were nine theorems (Theorems 5-13) involving capacity spaces that were given without proof. One of these theorems asserted that a LOTS (=linearly ordered topological space) with a capacity is metrizable. In [BL] this result is obtained as a corollary to the more general result.

*Theorem* 1. *A* GO-*space (=generalized ordered space) with a capacity has a* $G_\delta$-*diagonal and is perfect (=closed sets are* $G_\delta$-*sets).*

The question of what subspaces of a capacity space have a capacity naturally arises.

*Theorem* 2. *Let* Y *be a subspace of a capacity space* X. *If* Y *is either a regularly closed, or open subset of* X, *then* Y *is a capacity space.*

*Proof.* Let $\varepsilon$ be a capacity on X. If Y is a regularly closed subspace of X define

$$\eta(x,F) = \varepsilon(x, F \cup Cl(X-F))$$

for $x \in Y$ and F closed in Y.

If Y is an open subspace of X define

$$\eta(x,F) = \varepsilon(x,F \cup (X-F))$$

for $x \in Y$ and f closed in Y. In either case it is not difficult to verify that $\eta$ is a capacity in Y.

It will be shown later that closed subspaces of capacity spaces need not have a capacity. To solve the dense subspace problem we need the following definition.

*Definition.* A capacity for a space X is *faithful* if (c-5) $F_1, F_2 \in \mathcal{F}$ such that Int $F_1$ = Int $F_2$, then $\varepsilon(x,F_1) = \varepsilon(x,F_2)$.

*Theorem* 3. *If* Y *is a dense subspace of a faithful capacity space* X, *then* Y *is a faithful capacity space.*

*Proof.* Let $\varepsilon$ be a faithful capacity on X. Define $\eta(x,F) = \varepsilon(x,\mathrm{Cl}(F,X))$ where Cl(F,X) is the closure of F in X. It is routine to verify that $\eta$ is a faithful capacity on Y.

The condition (c-5) leads to the following question.

*Question* 1. Does there exist a capacity space that *does not* have a faithful capacity?

The following example gives a non-faithful capacity of [0,1] with the usual topology. Since this space is metric it also has a faithful capacity.

*Example.* For a closed set F in X = [0,1] with the usual topology let $\varepsilon(x,F) = d(x,X-F) \cdot m^*(F)$ where d is the usual metric on [0,1] and $m^*(F)$ is the outer Lebesgue measure of F.

It is straight forward to verify that this is a non-faithful capacity on [0,1].

In 1980 in another paper [$S_2$] by Scepin, the notion of a $\kappa$-metric on a completely regular space X is introduced. Let $\mathcal{C}$ be the class of all regularly closed subsets of X. Then a nonnegative real-valued function $\rho$ with domain $X \times \mathcal{C}$ is a $\kappa$-metric on X if

(k-1) $\rho(x,C) = 0$ if and only if $x \in C$,

(k-2) if $C_1$ and $C_2$ are in $\mathcal{C}$ and $C_1 \subset C_2$, then $\rho(x,C_1) \geq \rho(x,C_2)$

(k-3) if C is fixed, then $\rho$ is continuous in the first variable, and

(k-4) if $\{C_\alpha\}$ is a transfinite increasing collection of elements of $\mathcal{C}$, then $\rho(x,\mathrm{Cl}(\bigcup_\alpha\{C_\alpha\})) = \inf_\alpha\{\rho(x,C_\alpha)\}$.

In [$S_2$] proofs for the theorems in [$S_1$] are indicated except for Theorem 12 of [$S_1$] which is stated as an open problem in [$S_2$]. These proofs are given in the $\kappa$-metrizable setting rather than the capacity setting. This is due, perhaps, to the unproven statement that spaces with a capacity "are identical with $\kappa$-metrizable spaces," ([$S_2$], p. 411).

The following theorem indicates that this may not be true and if the answer to Question 1 is yes then the theorems in [$S_1$] do need to be addressed in a capacity space setting.

*Theorem* 4. *A completely regular space* X *has a faithful capacity if and only if* X *is* $\kappa$*-metrizable.*

*Proof.* Let X have a faithful capacity $\varepsilon$. For each

$C \in \mathcal{C}$ let

$$\rho(x,C) = \varepsilon(x,\mathrm{Cl}(X-C))$$

Conditions (k-1), (k-2), and (k-3) are readily obtainable. To see that (k-4) is true, let $\{C_\alpha\}$ be an increasing collection of elements of $\mathcal{C}$. Then $\rho(x,\mathrm{Cl}(\cup C_\alpha)) = \varepsilon(x,\mathrm{Cl}(X-\mathrm{Cl}(\cup C_\alpha)))$. On the other hand

$$\inf_\alpha\{\rho(x,C_\alpha)\} = \inf_\alpha\{\varepsilon(x,\mathrm{Cl}(X-C_\alpha)\} = \varepsilon(x,\cap\mathrm{Cl}(X-C_\alpha))$$

since $\{\mathrm{Cl}(X-C_\alpha)\}$ is a decreasing collection of closed sets. Since $\varepsilon$ is faithful capacity we must show

$$\mathrm{Int}(\mathrm{Cl}(X-\mathrm{Cl}(\cup C_\alpha))) = \mathrm{Int}(\cap\mathrm{Cl}(X-C_\alpha))$$

To this end let $z \in \mathrm{Int}(\mathrm{Cl}(X-\mathrm{Cl}(\cup C_\alpha)))$. Then there is an open set U containing z such that $U \subset \mathrm{Cl}(X-\mathrm{Cl}(\cup C_\alpha))$. Since each $C_\alpha$ is regularly closed and $\{C_\alpha\}$ is an increasing collection, $U \cap C_\alpha = \emptyset$ for each $\alpha$. Thus $U \subset X-C_\alpha \subset \mathrm{Cl}(X-C_\alpha)$. Hence $U \subset \cap\mathrm{Cl}(X-C_\alpha)$ and it follows that $z \in \mathrm{Int}(\cap\mathrm{Cl}(X-C_\alpha))$.

If $z \in \mathrm{Int}(\cap\mathrm{Cl}(X-C_\alpha))$ then there is an open set U containing z such that $U \subset \cap\mathrm{Cl}(X-C_\alpha)$. Hence for each $\alpha$, $U \subset \mathrm{Cl}(X-C_\alpha)$ and, since $C_\alpha$ is regularly closed, $U \cap C_\alpha = \emptyset$. Thus $U \cap (\cup C_\alpha) = \emptyset$. It follows that $U \cap \mathrm{Cl}(\cup C_\alpha) = \emptyset$. Thus $U \subset X-\mathrm{Cl}(\cup C_\alpha)$. Hence $z \in \mathrm{Int}(\mathrm{Cl}(X-\mathrm{Cl}(\cup C_\alpha)))$. Hence $\mathrm{Int}(\mathrm{Cl}(X-\mathrm{Cl}(\cup C_\alpha))) = \mathrm{Int}(\cap\mathrm{Cl}(X-C_\alpha))$ and, since $\varepsilon$ is faithful, we have $\varepsilon(x,\mathrm{Cl}(X-\mathrm{Cl}(\cup C_\alpha))) = \varepsilon(x,\cap\mathrm{Cl}(X-C_\alpha))$ from which it follows that $\rho(x,\mathrm{Cl}(\cup C_\alpha)) = \inf\{\rho(x,C_\alpha)\}$.

Conversely, let $\rho$ be a $\kappa$-metric on X. For each $x \in X$ and closed set F in X let $\varepsilon(x,F) = \rho(x,\mathrm{Cl}(X-F))$. Notice that $\varepsilon$ is well-defined since if F is closed then $\mathrm{Cl}(X-F)$ is regularly closed. Conditions (c-1), (c-2) and (c-3) are readily obtainable.

To see that (c-4) is true let $\{F_\alpha\}$ be a decreasing sequence of closed sets. Then

$$\varepsilon(x,\cap F_\alpha) = \rho(x,\mathrm{Cl}(X-\cap F_\alpha))$$

and

$$\inf_\alpha\{\varepsilon(x,F_\alpha)\} = \inf_\alpha\{\rho(x,\mathrm{Cl}(X-F_\alpha)\} = \rho(x,\mathrm{Cl}(\cup \mathrm{Cl}(X-F_\alpha))).$$

Thus, it must be shown that

$$\mathrm{Cl}(X-\cap F_\alpha) = \mathrm{Cl}(\cup \mathrm{Cl}(X-F_\alpha)).$$

To this end let $z \in \mathrm{Cl}(X-\cap F_\alpha) = \mathrm{Cl}(\cup(X-F_\alpha))$. For each open set U containing z, it follows that $U \cap (\cup(X-F_\alpha)) = \emptyset$. Hence

$$z \in \mathrm{Cl}(\cup(X-F_\alpha)) \subseteq \mathrm{Cl}(\cup \mathrm{Cl}(X-F_\alpha)).$$

Let $z \in \mathrm{Cl}(\cup \mathrm{Cl}(X-F_\alpha))$. Then for each open set U containing z there is a member M(U) of the well-ordered indexing set to which $\alpha$ belongs, such that if M(U) precedes $\alpha$ in the well-ordering, then $U \cap \mathrm{Cl}(X-\cap F_\alpha) \neq 0$ and $z \in \mathrm{Cl}(X-\cap F_\alpha)$. Hence $\varepsilon$ is a capacity for X.

If $F_1$ and $F_2$ are closed sets such that Int $F_1$ = Int $F_2$, then $\mathrm{Cl}(X-F_1) = \mathrm{Cl}(X-F_2)$. Thus, if $x \in X$,

$$\rho(x,\mathrm{Cl}(X-F_1)) = \rho(x,\mathrm{Cl}(X-F_2)).$$

From this it follows that

$$\varepsilon(x,F_1) = \varepsilon(x,F_2)$$

and $\varepsilon$ is a faithful capacity on X.

In [$S_2$] it is shown that the product of $\kappa$-metrizable spaces is a $\kappa$-metrizable space. This fact is used in the next example.

*Example.* There is a $\kappa$-metrizable space X with a closed subspace Y that does not have a capacity.

Let Z be an uncountable subset of [0,1] whose only compact (with regard to the usual topology) subsets are countable (see [K]). Let Y = Cl(Z,[0,1]. Topologize Y with a finer topology $\tau$ than the relative Euclidean topology by letting points of Y-Z be discrete. It follows that $(Y,\tau)$ is a quasi-developable, [$B_1$], GO-space which is not metrizable. If $(Y,\tau)$ had a capacity, then, by Theorem 1, $(Y,\tau)$ would be perfect and, hence, developable [$B_1$]. Since $(Y,\tau)$ is a GO-space it would be metrizable. From this contradiction it follows that $(Y,\tau)$ cannot have a capacity.

It is not difficult to prove that $(Y,\tau)$ is a Lindelöf space and, hence, realcompact. Thus $(Y,\tau)$ can be closed embedded in a product of real lines. Let $X = \prod_\alpha R_\alpha$ (where each $R_\alpha$ is a copy of the real line) contain a homeomorphic copy of $(Y,\tau)$ as a closed subset. Since each $R_\alpha$ is $\kappa$-metrizable it follows that X is $\kappa$-metrizable [$S_2$, pg. 408]. Hence closed subspaces of $\kappa$-metrizable spaces need not have a capacity.

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