INEQUIVALENT EMBEDDINGS AND PRIME ENDS

by

JOHN C. MAYER
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1. Introduction

In [B-M] Brechner and Mayer show that equivalent embeddings of a nonseparating plane continuum have the same prime end structure (Theorem 2.11). Though not explicitly stated, this fact has been used previously in the literature. For instance, M. Smith [Sm] and W. Lewis [L] have independently shown that there are uncountably many inequivalent embeddings of the pseudo arc into the plane. This result was achieved by exploiting different prime end structures (directly in Lewis' case, indirectly, in terms of differing accessibility of composants by Smith) to distinguish different embeddings.

In this paper we show that the converse of Brechner and Mayer's theorem is false: there are inequivalent embeddings of a nonseparating continuum into the plane that have the same prime end structure, and indeed, that have the same set of accessible points. The following theorems stand in partial contrast to the methods of Smith and Lewis:

1.1 Theorem. There exist uncountably many inequivalent embeddings of the $\sin \frac{1}{x}$ continuum into the plane with the same prime end structure and the same set of accessible points.

1.2 Theorem. There exist uncountably many inequivalent embeddings of the Knaster U-continuum (bucket handle)
into the plane with the same prime end structure. Moreover, the set of accessible points in each of these embeddings is exactly the composant of the U-continuum that contains the endpoint of the U-continuum.

In Section 2, we show how to construct uncountably many embeddings of the continuum formed by the sin l/x curve plus its limit segment. Thereafter, we show these embeddings have the same prime end structure and the same set of accessible points, but that any two are inequivalent, proving Theorem 1.1.

In Section 3, we proceed similarly to prove Theorem 1.2, showing the Knaster U-continuum also has uncountably many inequivalent embeddings with the same prime end structure and the same set of accessible points.

In Section 4, we indicate how Theorem 1.2 can be extended to each of the uncountable class of U-type Knaster continua identified by W. T. Watkins [W].

This paper, together with [M-1] and [M-2], comprise the major portion of the author's dissertation. Section 3, in particular, has benefited from numerous discussions with the author's advisor, Beverly Brechner.

2. The Sin l/X Continuum

The standard embedding of the sin l/x continuum (Figure 1) consists of a ray R, the graph of (0,1] under the function \( y = \sin \frac{1}{x} \) in the xy-plane, plus the limit segment \([p,q]\), the interval \([-1,1]\) on the y-axis. The ray R consists of a number of loops, where a loop is a segment
of \( R \) with exactly one peak and one trough in the standard embedding. For simplicity, we will fix the set of loops as the segments of \( R \) between alternate successive points of zero amplitude in order from 1 to 0 on the \( x \)-axis. The endpoints of loops limit on \((0,0)\) in the standard embedding, so on the "midpoint" (or, more precisely, some interior point) of \([p,q]\) in any other embedding. Endpoint \( p \) of limit segment \([p,q]\) is the limit point of points on \( R \) selected from successive troughs, and endpoint \( q \) is the limit point of points selected from successive peaks. We suppose \( R \) to be coordinatized by the function \( g: [0,\infty) \rightarrow \mathbb{R} \) so that the endpoint of \( R \) is \( g(0) \) and for \( x \in (0,1] \), odd positive integers correspond to \( \{(x,1): \sin \frac{1}{x} = 1\} \), even positive integers correspond to \( \{(x,-1): \sin \frac{1}{x} = -1\} \), and fractions with denominator 2 and odd numerator greater than 1 correspond to \( \{(x,0): x = 1/(\pi n)\} \), for positive integers \( n \). This embedding, denoted \( K \), and its coordinatization by \( g \) are illustrated in Figure 1.

An embedding \( e: X \rightarrow \mathbb{R}^2 \) of a continuum \( X \) is a homeomorphism into the plane; however, we shall somewhat loosely suppress reference to an embedding function and refer to the image in the plane of a continuum \( X \) as the embedding of \( X \).

An embedding of the \( \sin \frac{1}{x} \) continuum can be described in terms of how the limit segment and ray is embedded in the plane. We will continue to designate the image of \( R \) \([p,q]\), or a point \( x \) in the \( \sin \frac{1}{x} \) continuum as \( R \) \([p,q]\), or \( x \), suppressing reference to any particular embedding function.
2.1 Embeddings of the sin $1/x$ continuum. Let $[p,q]$ be the vertical line segment between $(0,1)$ and $(0,-1)$ on the y-axis in the xy-plane, identifying $(0,1)$ with $q$ and $(0,-1)$ with $p$. For each integer $i > 1$, let $L_i(R_i)$ be the vertical segment between $(-1/i,-1)$ and $(-1/i,1)$ (between $(1/i,-1)$ and $(1/i,1)$). The upper endpoint of $L_i(R_i)$ is $(-1/i,1)$ ($(1/i,1)$), and the lower endpoint of $L_i(R_i)$ is $(-1/i,-1)$ ($(1/i,-1)$). For each $i$, connect the upper endpoints of $L_i$ and $R_i$ by a semicircle, called an upper semicircle, centered at $q$ and lying, except for endpoints, above the line $y = 1$. We will connect the lower endpoints of the $L_i$ and $R_i$'s in some specified order to produce a ray converging to $[p,q]$.

A subschema is a set of directions for connecting a finite number of lower endpoints of $L_i$'s and $R_i$'s by semicircles lying, except for endpoints, below the line $y = -1$, called lower semicircles. In a subschema, connect $L_k$ to $L_{k+r}$ means that we connect the lower endpoints of $L_k$ and $L_{k+r}$ by a semicircle centered midway between these endpoints on the line $y = -1$, and lying, except for these endpoints, below the line $y = -1$. We similarly define connect $R_k$ to $R_{k+r}$. By start at $L_k$ we mean that the subschema begins with a semicircle from $L_k$ to some $L_{k+r}$. If a subschema begins with a semicircle from $L_k$ to $L_{k+r}$, then the last connection instruction will be "connect $R_{k+1}$ to $R_{k+r+1}". We say end at $L_{k+r+1}$ to indicate that in executing the subschema, the connecting lower semicircles, together with the $L_i$'s and $R_i$'s connected, together with the upper
semicircles previously constructed connecting $L_{k+1}$ to
$R_{k+1}, \ldots, L_{k+r+1}$ to $R_{k+r+1}$' comprise an arc from the lower
endpoint of $L_k$ to the lower endpoint of $L_{k+r+1}$'. The follow­
ing is a countably infinite list of subschemata:

$S_0$: Start at $L_k$;
Connect $L_k$ to $L_{k+1}$; connect $R_{k+1}$ to $R_{k+2}$;
End at $L_{k+2}$.

$S_1$: Start at $L_k$;
Connect: $L_k$ to $L_{k+1}$; $L_{k+2}$ to $L_{k+1}$;
$R_{k+2}$ to $R_{k+4}$;
End at $L_{k+4}$.

$S_2$: Start at $L_k$;
Connect: $L_k$ to $L_{k+2}$; $L_{k+4}$ to $L_{k+1}$;
$L_{k+2}$ to $L_{k+1}$; $R_{k+4}$ to $R_{k+6}$;
End at $L_{k+6}$.

$S_n$: Start at $L_k$;
Connect: $L_k$ to $L_{k+2n+1}$; $R_{k+2n+1}$ to $R_{k+2n}$; $\cdots$
$L_{k+2n-1}$ to $L_{k+2n-1}$; $R_{k+2n-1}$ to $R_{k+2n-2}$; $\cdots$
$L_{k+2}$ to $L_{k+1}$; $R_{k+2}$ to $R_{k+2n+2}$;
End at $L_{k+2n+2}$.

By executing $S_n$, the lower endpoints of $L_k$ through
$L_{k+2n+1}$ inclusive are connected in pairs by non-intersecting
semicircles: one connects $L_k$ to $L_{k+2n+1}$ and the others con­
nnect adjacent pairs of $L_1$'s. Similarly, $R_{k+1}$ is connected
to $R_{k+2n+2}$ by one semicircle and the intervening $R_1$'s are
connected in adjacent pairs by the remaining semicircles. The result is that we have constructed an arc from the lower endpoint of $L_k$ to the lower endpoint of $L_{k+2n+2}$ comprised of vertical segments, semicircles above $y = 1$, and semicircles below $y = -1$. By subschema $S_n$ we will, somewhat ambiguously, refer to the directions for making connections and the resulting arc from $L_k$ to $L_{k+2n+2}$.

Let $N = \{a_1, a_2, a_3, \ldots\}$ be an infinite sequence of nonnegative integers. A schema $P = \{S_{a_1}, S_{a_2}, S_{a_3}, \ldots\}$ is an infinite sequence of subschemata $S_{a_i}$ arranged so that $S_{a_i}$ begins where $S_{a_i}$ ends. That is, if $k = 1 + 2(a_1+1) + 2(a_2+1) + \cdots + 2(a_{i}+1)$, then $S_{a_i+1}$ begins at $L_k$ and ends at $L_{k+2(a_i+1)}$. Executing each subschema of $P_a$ in order results in a ray $R_a$ which contains every $L_i$ and $R_i$ and converges to $[p,q]$.

We coordinatize $R_a$ by identifying the lower endpoint of $R_1$ with 0, the midpoint of each semicircle above $y = 1$ with an odd integer, and the midpoint of each semicircle below $y = -1$ with an even integer, in an order-preserving manner on $R_a$. The order for the correspondence is, in fact, built into the order in which the connection instructions of $S_n$ are stated. Figure 2 illustrates the embedding $M_0$ of the sin $1/x$ continuum produced by executing the schema $P_0 = \{S_0, S_0, S_0, \ldots\}$. Figure 3 illustrates an embedding $M_a$ produced by executing the schema $P_a = \{S_2, S_1, \ldots\}$.

Note that for $n > 0$, the first (last) semicircle of schema $S_n$ "skips over" $2n$ vertical segments to connect $L_k$.
to $L_{k+2n+1}$ ($R_{k+1}$ to $R_{k+2n+2}$). The segments skipped over are connected in adjacent pairs on each side of $[p,q]$. We call the first (last) semicircle of $S_n$, for $n \geq 0$, a bend toward $p$; we call the $n$ intervening semicircles on each side of $[p,q]$ bends away from $p$. By an outer loop we mean a loop of $R_a$ which contains a bend toward $p$, and by an inner loop we mean a loop of $R_a$ which contains a bend away from $p$.

Observe that in $E^2$ outer loops locally separate inner loops from $p$. By this we mean that for a sufficiently small $\epsilon$-ball $S(p,\epsilon)$ about $p$, each arc in $S(p,\epsilon)$ from $p$ to an inner loop must meet some outer loop. (In fact, it must meet an outer loop of the subschema that contains the inner loop.)

2.2 Definition. Let $N$ represent the set of nonnegative integers. Two infinite sequences $A$ and $B$ selected from $N$ (with replacement) are distinct iff after removing any finite (possibly null) initial subsequence from $A$ and any finite (possibly null) initial subsequence from $B$, the remaining infinite sequences $A'$ and $B'$ are not identical. For example, $\{1,3,5,\ldots\}$ and $\{2,4,6,\ldots\}$ are distinct; $\{0,0,0,\ldots\}$ and $\{1,1,1,\ldots\}$ are distinct; $\{1,2,3,\ldots\}$ and $\{4,5,6,\ldots\}$ are not distinct. The definition of distinct sequences generalizes to infinite sequences selected from any set indexed by $N$.

2.3 Lemma. There exist uncountably many distinct infinite sequences selected from $N$. 
2.4 Lemma. There exist uncountably many distinct schemata for embedding the \( \sin \frac{1}{x} \) continuum.

Proof. We use the notation of Section 2.1 and Definition 2.2. Let \( \{S_0, S_1, S_2, \ldots\} \) be the set of subschemata and let \( \{N_a\}_{a \in A} \) be the set of distinct infinite sequences selected from \( N \). Suppose that \( N_a = \{a_1, a_2, a_3, \ldots\} \) and let \( P_a = \{S_{a_1}, S_{a_2}, S_{a_3}, \ldots\} \) be a schema for embedding the \( \sin \frac{1}{x} \) continuum. It follows that \( \{P_a\}_{a \in A} \) is uncountable, and that for all \( a \neq b \in A, P_a \) and \( P_b \) are distinct lists of subschemata.

2.5 The prime end structures of \( K \) and \( M_0 \). Before proceeding to construct our uncountably many embeddings of the \( \sin \frac{1}{x} \) continuum, we illustrate some of the concepts involved in prime end theory by applying them to \( K \) and \( M_0 \). Definitions and further references may be found in [Br].

Prime ends are a way of looking at and classifying the approaches to the boundary of a simply connected domain with nondegenerate boundary. The complement in \( S^2 \) of a non-separating nondegenerate plane continuum \( X \), denoted \( S^2 - X \), is a simply connected domain. While \( E^2 - X \) is not simply connected, as \( E^2 \cup \{\omega\} \), the one-point compactification of \( E^2 \), is \( S^2 \), we can refer to the prime end structure of \( E^2 - X \) by associating it with the prime end structure of \( S^2 - X \), where the embedding of \( X \) misses the point at infinity.

A prime end of \( E^2 - X \) is defined by a chain of crosscuts converging to a point of \( X \), where a crosscut is an open arc in \( E^2 - X \) whose endpoints lie in \( X \). If \( Q \) is a crosscut of \( E^2 - X \), then \( Q \cup X \) separates \( E^2 \). A sequence of crosscuts
\( \{Q_i\} \) is a chain provided that \( Q_i \) converges to a point, that no two crosscuts have a common endpoint, and that \( Q_i \) separates \( Q_{i-1} \) and \( Q_{i+1} \) in \( E^2 - X \). So, for example, the prime ends \( E \) and \( F \) of \( E^2 - K \) in Figure 1 are defined by chains of crosscuts \( \{Q_i\}_{i=1}^{\infty} \) and \( \{Q'_i\}_{i=1}^{\infty} \), respectively, while in Figure 2, \( \{T_i\}_{i=1}^{\infty} \) defines prime end \( H \) of \( E^2 - M_0 \).

The *impression* of a prime end \( E \), denoted \( I(E) \), is the intersection of the closures of the bounded domains cut off by the crosscuts in a chain defining \( E \). For example, in Figure 1 it can be noted that \( I(E) = [p,q] = I(F) \), while in Figure 2, \( I(H) = [p,q] \).

The *set of principal points* of a prime end \( E \), denoted \( P(E) \), is the collection of all points in \( X \) to which some chain of crosscuts defining \( E \) converges. For example, in Figure 1, \( P(E) = \{p\} \), and \( P(F) = \{q\} \), while in Figure 2, \( P(H) = \{p\} \).

A prime end \( E \) is of the *first kind* if \( I(E) = P(E) \), both degenerate, of the *second kind* if \( I(E) \neq P(E) \), only \( P(E) \) degenerate, of the *third kind* if \( I(E) = P(E) \), both nondegenerate, and of the *fourth kind* if \( I(E) \neq P(E) \), both nondegenerate. It can be shown that \( P(E) \subseteq I(E) \) in any case, and that both are continua in \( X \). Thus prime ends \( E, F, \) and \( H \) of Figures 1 and 2 are all of the second kind. Any other prime end \( G \), of either \( E^2 - K \) or \( E^2 - M_0 \), will be of the first kind, or trivial. Thus we can say that the prime end structure of \( E^2 - K \) consists of two prime ends of the second kind and all other prime ends trivial.

A more precise description of prime end structure is afforded by the notion of a *C-map*. A C-map \( \phi \) is a
homeomorphism of $S^2 - X$ onto $\text{Ext } B$, where $\text{Ext } B$ is the complementary domain of the unit disk $B$ in $S^2$ which contains the point at infinity, which (map) satisfies the conditions:

1. if $Q$ is a crosscut of $S^2 - X$, then $\phi(Q)$ is a crosscut of $\text{Ext } B$, and

2. the endpoints of images of crosscuts are dense in $\text{Bd } B$, the boundary of $B$.

If we require, as we may, that $\phi$ take the point at infinity in $S^2 - X$ to the point at infinity in $\text{Ext } B$, then we can regard the restriction of $\phi$ to $S^2 - \{ \infty \} = E^2$ as a C-map also.

Suppose a chain of crosscuts defines a prime end $E$ of $E^2 - X$. Then the images of the crosscuts under a C-map $\phi$ will converge to a single point $e$ in $\text{Bd } B$. We say $e$ corresponds to $E$. In fact, there is a one-to-one correspondence between the prime ends of $E^2 - X$ and the points of $\text{Bd } B$. For example, in Figure 1, points $e$ and $f$ in $\text{Bd } B$ correspond to prime ends $E$ and $F$, respectively. In Figure 2, $h$ corresponds to $H$. There is no homeomorphism $g: \text{Bd } B \leftrightarrow \text{Bd } B$ such that both $g(e)$ and $g(f)$ are points corresponding to prime ends of the second kind, since $h$ corresponds to the only such prime end in $E^2 - M_0$. Hence the prime end structures of $K$ and $M_0$ are not identical [B-M, Definition 2.10]. Consequently, $K$ and $M_0$ are inequivalently embedded [B-M, Theorem 2.11].

In the general result which follows, each embedding of the sin $1/x$ continuum will have the same prime end structure as $M_0$. Note that the accessible points of $M_0$ are the ray $R$ and point $p$ of $[p, q]$. A point of a plane
continuum $X$ is accessible if it can be reached by a half open arc in the complement whose closure adds exactly that point. Such a half open arc is called an \textit{endarc}. Each of our uncountably many embeddings of the $\sin \frac{1}{x}$ continuum will have the same set of accessible points as $M_0$. The significance of this result is the contrast it provides to the usual procedure for producing inequivalent embeddings of a plane continuum: produce embeddings with different points accessible. Such a procedure is a sufficient, but not a necessary, condition for producing inequivalent embeddings of a continuum in $\mathbb{E}^2$.

2.6 \textit{Proof of Theorem 1.1.} Let $\{P_a\}_{a \in A}$ be the uncountable set of distinct schemata demonstrated in Lemma 2.4. For each $a \in A$, let $M_a$ be the embedding of the $\sin \frac{1}{x}$ continuum according to schema $P_a$. We claim the prime end structures of $M_a$ and $M_b$ for all $a \neq b \in A$ are identical. Any chain of crosscuts converging to $p$ in $M_a$ ($M_b$) defines a prime end $E_a$ ($E_b$) of the second kind, for which $[p,q]$ is the impression and $p$ is the only principal point. Any other prime end of $M_a$ ($M_b$) is trivial. Moreover, the set of accessible points of $M_a$ ($M_b$) is the ray $R_a$ ($R_b$) and the point $p$ of $[p,q]$. Hence $M_a$ and $M_b$ have the same prime end structure and the same set of accessible points.

We claim that $M_a$ and $M_b$ are inequivalent embeddings. By way of contradiction, suppose that $M_a$ and $M_b$ are equivalently embedded. Then there is a homeomorphism $h: \mathbb{E}^2 \to \mathbb{E}^2$ onto the plane such that $h(M_b) = M_a$. Then $h(R_b) = R_a$, and $h$ must be order-preserving on $R_b$. 
In $M_b$ let $\{p_j\}_{j=1}^\infty$ be those even integer points on $R_b$ which lie on outer loops; that is, those integer points of $R_b$ for which a sequence of crosscuts $\{Q_j\}_{j=1}^\infty$, each $Q_j$ joining $p_j$ to $p$, can be chosen so that $\text{diam } Q_j \to 0$ as $j \to \infty$.

(These points are on the bends toward $p$ in each subschema. Note that this sequence of crosscuts is not a chain.) For odd $j$, each pair $p_j, p_{j+1}$ of outer loop points is separated on $R_b$ by a finite set $\{p_j(1), p_j(2), \ldots, p_j(n_j)\}$ of even integer points of $R_b$ which lie on inner loops. (These points are on the bends away from $p$ of each subschema.) For even $j$, $p_j$ and $p_{j+1}$ are in succession on $R_b$, with no inner loops between them. Note that any crosscut joining $p_j(1)$ to $p$ has diameter no less than that of $[p,q]$. For odd $j$, let $A_j = \{p_j, p_j(1), p_j(2), \ldots, p_j(n_j), p_{j+1}\}$ be the sequence of even integer points in order on $R_b$ described in the preceding paragraph. Consider $h(A_j)$. Since $\text{diam } Q_j \to 0$, $\text{diam } h(Q_j) \to 0$, as well. Hence outer loop points $p_j, p_{j+1}$ cannot infinitely often be carried to loop ends of $R_a$ in the neighborhood of $p$ which contain inner loop points. Similarly, inner loop points of $R_b$ cannot infinitely often be carried onto loop ends of $R_a$ that contain outer loop points. As the even integer points converge to $p$, and $h$ is order-preserving on $R_b$, there is a positive odd integer $n$, such that for all $j \geq n$, the set $A_j$, consisting of two outer loop points and $n_j$ inner loop points between them, must be carried into a set of loop ends of $R_a$ in the close neighborhood of $p$ consisting of two outer loop ends and $n_j$ inner loop ends between them.
Each loop end of $R_a$ in the neighborhood of $p$ contains an even integer point of $R_a$. For simplicity we may assume that $h$ carries, from $P_n$ onward, even integer points of $R_b$ to even integer points of $R_a$, one-to-one and order-preserving.

If $\{p'_k\}_{k=1}^\infty$ is the set of even integer points of $R_a$ on outer loops, then our preceding remarks require that there be a positive odd integer $m$ such that there is a set of even integer points $B_m = \{p'_m, p'_m(1), p'_m(2), \ldots, p'_m(n), p'_m+n\}$ that corresponds one-to-one to the set $A_n$, including $p'_m(i)$ being an inner loop point. Thereafter, $B_{m+2}$ corresponds one-to-one to $A_{n+2}$, $B_{m+4}$ to $A_{n+4}$, etc.

Each subschema contains only two outer loops. Hence $A_n$ and $B_m$ each correspond to a listing of the even integer points in a single subschema, $S_{a(n+1)/2}$ and $S_{b(m+1)/2}$, respectively. Likewise for $A_{n+2}$ and $B_{m+2}$, etc. This implies that in $P_a$ and $P_b$, $S_{a(n+r)/2} = S_{b(m+r)/2}$, for all odd $r > 1$.

Thus $P_a$ is identical to $P_b$ after the removal of some finite initial subsequence from one and some finite initial subsequence from the other. This contradicts our choice of $P_a$ and $P_b$ distinct.

Therefore, no such homeomorphism $h$ can exist. Hence, $M_a$ and $M_b$ are inequivalently embedded, but have the same prime end structure and same set of accessible points.

2.7 Remark. The uncountable class above does not exhaust the inequivalent embeddings of the sin $1/x$ continuum. All the embeddings we have considered have the prime end structure of Figure 2. There are also uncouthably many
inequivalent embeddings of the sin $1/x$ continuum with the prime end structure of Figure 1, and every point accessible. Additional embeddings with the prime end structure of Figure 1 are illustrated in Figure 4. In Figure 5(a) we show an embedding with a prime end $G$ defined by any chain of cross-cuts converging to some point of $[p,q]$; for example, chain $\{P_i\}_{i=1}^{\infty}$. Note that $I(G) = P(G) = [p,q]$. Figure 5(b) is another embedding with the same prime end structure. In these embeddings, only points of the ray $R$ are accessible.

3. The Knaster U-Continuum

The Knaster U-continuum $K$ can be represented abstractly as the inverse limit system $K = \lim_{i \to \infty} (I_i, f_i)_{i=1}^{\infty}$, where $I_i = [0,2^i]$ and $f_i: [0,2^{i+1}] \to [0,2^i]$ is the rooftop function defined by

$$f_i(x) = \begin{cases} x, & \text{if } 0 \leq x < 2^i \\ 2^i + 1 - x, & \text{if } 2^i \leq x \leq 2^{i+1} \end{cases}$$

This representation, used by Watkins in [W], has the advantage of providing a convenient coordinatization of the endpoint composant of $K$. The endpoint of $K$ is $(0,0,0,\cdots)$, and the composant of the endpoint is that set of points in $K$ whose coordinates, from some $i^{th}$ coordinate onward, are constant.

3.1 Standard embedding $K_0$ of $K$ in $E^2$. We construct the standard embedding $K_0$ of $K$ in $E^2$ below. Let $\mathcal{C}$ be the middle third Cantor set constructed on the unit segment $I$ between $(0,0)$ and $(1,0)$ in the xy-plane. By a deleted interval we mean a complementary interval of $I-\mathcal{C}$. By a
subinterval of $C$ we mean a first or last third interval at the $k$th stage in the construction of $C$ intersected with $C$, that is any $[m/3^k, (m+1)/3^k] \cap C$ with non-empty interior in $C$.

Connect the points of $[0,1/3] \cap C$ to the points of $[2/3,1] \cap C$ by a Cantor set of semicircles lying above the x-axis (except for their endpoints in $C$) and centered at $(1/2,0)$. For each $k = 2,3,\cdots$, connect the points of sub-interval $[6/3^k, 7/3^k] \cap C = D_{k-1,2}$ with the points of sub-interval $[8/3^k, 9/3^k] \cap C = D_{k-1,1}$ by a Cantor set of semicircles lying below the x-axis (except for their endpoints in $C$) and centered at the point midway between these sub-intervals. The union of all semicircles thus constructed is a continuum $K_0 \subset E^2$, with we call the standard embedding of $K$. In fact it is Knaster's original construction. See Figure 6.

In this and subsequent sections, semicircles lying above the x-axis, except for their endpoints, we term upper semicircles, and semicircles lying below the x-axis, except for their endpoints, we term lower semicircles. Concentric semicircles are said to be parallel semicircles. We extend this term to apply to a collection of arcs which are piecewise composed of concentric semicircles. Among a closed collection of concentric semicircles, we call that with largest radius an outer semicircle, and that with smallest radius an inner semicircle.

3.1.1 Topological equivalence of $K$ and $K_0$. We claim that $K$ and $K_0$ are homeomorphic (so that $K_0$ is indeed an
embedding of $K$). First note that there is a ray $C_0 \subset K_0$ whose endpoint is $(0,0)$ and which includes $(1,0)$ and the endpoints of every deleted interval. Now $C_0$ is the composant of $(0,0)$ in $K_0$. We will coordinatize $C_0$ by $[0,\infty)$ in such a way that $0$ corresponds to $(0,0)$, each odd positive integer corresponds to the midpoint of an upper semicircle, and each even positive integer corresponds to the midpoint of a lower semicircle. Not all upper and lower semicircles are involved, but only those in $C_0$. We will also show that there is a retraction $r_k: K_0 \rightarrow [0,2^k] \subset C_0$, for each $k = 1,2,3,\ldots$, so that

1. $r_k$ moves no point more than $1/2^k$, and
2. $r_k = f_k \circ r_{k+1}$.

It then follows that $\lim r_k = r: K_0 \rightarrow K$ is a homeomorphism.

For each $k = 1,2,3,\ldots$, let $a_k$ denote the midpoint of the outer semicircle in the collection joining $D_{k,1}$ to $D_{k,2}$, and let $d_k$ denote the midpoint of the inner semicircle. Let $L_k$ be the vertical segment joining $a_k$ to $d_k$. Note that $L_k \cap K_0$ is a cut of $K_0$. That is, $L_k \cap K_0$ is an irreducible separator of $K_0$. Necessarily, a cut of $K_0$ is a Cantor set.

For nonnegative reals $x$ and $y$, $x < y$ denotes, in the usual order, that $x$ is less than $y$. For any ray $R$, for $x,y \in R$, $x < y$ denotes that $f^{-1}(x) < f^{-1}(y)$ for any coordinatization $f$ of $R$ by $[0,\infty)$. For subsets $A,B$ of a ray $R$ (or of $[0,\infty)$), $A < B$ denotes that $x < y$ for all $x \in A$ and $y \in B$. The symbols $>$, $\leq$, and $\geq$ are similarly defined.

For any ray $R$, for any $x < y \in R$, let $[x,y]$ denote that interval in $R$ between $x$ and $y$, inclusive; that is, $\{z \in R | x \leq z \leq y\}$. 
If $S$ is a continuum separating $E^2$ into exactly two components, we let $\text{Int}(S)$ denote the bounded complementary domain of $S$ and we let $\text{Ext}(S)$ denote the unbounded complementary domain of $S$.

Let $0$ denote the point $(0,0) \in C_0$. Note that $L_k \cup [a_k, d_k]$ is a simple closed curve. Define the following closed subsets of $K_0$ which meet only in the cut $L_k$:

$$[0, L_k] = \text{Cl}(\text{Ext}(L_k \cup [a_k, d_k])) \cap K_0$$
$$[L_k, L_{k+1}] = \text{Cl}(\text{Int}(L_k \cup [a_k, d_k])) \cap K_0$$

The notation is meant to suggest that the first set contains $0$, the second set contains the next cut, and that the cut $L_k$ lies between them.

Observe that $[0, a_k] \subset [0, L_k]$ and that $[a_k, a_{k+1}] \subset [L_k, L_{k+1}]$. It is not hard to see that we should coordinatize $C_0$ by identifying $2^k$ with $a_k$ and $2^{k+1} + 2^k$ with $d_k$, and that we can extend the coordinatization to intervening points in such a way that a retraction $r_k$ can be defined satisfying (1) and (2). We omit details as we shall carry out a similar process in Section 3.2.5 for more complicated embeddings. Note, however, that the order of points on $C_0$ is $0 < a_1 < a_2 < d_1 < a_3 < d_2 < \cdots < a_k < d_{k-1} < a_{k+1} < \cdots$.

### 3.2 Embeddings of the Knaster U-continuum.

For each sequence $N_a = \{a_1, a_2, \cdots\}$ of integers such that $a_k \geq 2$, we will define an inverse limit system $\hat{K}_a = \lim_{\leftarrow} \{K_a', g_k\}_{k=1}^\infty$, and we will construct what we call the natural embedding $K_a$ of $\hat{K}_a$ in $E^2$. 
Let \( n_1 = 1 \), and let \( n_k = 1 + \sum_{j=1}^{k-1} a_j \), for \( k = 2, 3, \ldots \).

For \( k = 1, 2, \ldots \), let \( A_k = [0, 2^{n_k}] \). Henceforth, unless otherwise stated, let \( k \) range over \( 1, 2, 3, \ldots \). Recall the definition of the rooftop functions \( f_k: [0, 2^{k+1}] \rightarrow [0, 2^k] \), and define \( g_k: A_{k+1} \rightarrow A_k \) by \( g_k = f_{n_k} \circ f_{n_{k+1}} \circ \cdots \circ f_{n_{k+1}^{k-1}} \).

Since each \( g_k \) is a composition of the next \( a_k \) unused \( f_j \)'s, it is clear that \( R_a \) is homeomorphic to \( K \).

We define the embedding \( K_a \) of \( R_a \) in \( E^2 \) below, coordinate the composant \( C_a \) of the endpoint of \( K_a \), and at the same time construct small retractions \( r_k \) from \( K_a \) onto subarcs of \( C_a \) that correspond to the factor spaces of \( \hat{K}_a \) so that the retractions commute with the \( g_k \)'s. Thus we show that \( K_a \) and \( \hat{K}_a \) are homeomorphic. We also develop notation and concepts which will be used in subsequent sections to show that there are uncountably many inequivalent embeddings of \( K \) in \( E^2 \), each the natural embedding of an inverse limit system based upon a sequence of integers like \( N_a \). It turns out that if two sequences of integers are distinct, the natural embeddings of the corresponding inverse limit systems are inequivalent.

### 3.2.1 Constructing the embedding \( K_a \) of \( \hat{K}_a \) in \( E^2 \)

Let \( C \) be the middle third Cantor set on the unit segment \( I = [0, 1] \) from \((0, 0)\) to \((1, 0)\) in the \( xy \)-plane. Connect points of \([0, 1/3] \cap C\) with points of \([2/3, 1] \cap C\) by a Cantor set of upper semicircles centered at \((1/2, 0)\). We connect the points of certain subintervals of \( C \) by Cantor sets of lower semicircles by executing the recursive procedure below for each \( a_k \in N_a \).
Subintervals of \( C \) are said to be equal provided that the corresponding intervals of \( I \) have equal length. Initially, let \( D_1^1 = C \cap [0,1/3] \) and \( D_1^2 = C \cap [2/3,1] \) be equal subintervals of \( C \). Iterate for each \( a_k \in N_a \) the following steps:

1. Assume \( D_1^k > D_2^k \) are equal subintervals of \( C \), as yet unconnected by lower semicircles. Let \( m_k = 2^{a_k-1} \).

2. In \( D_1^k \) identify \( 2m_k \) equal subintervals of \( C \):
\[
D_{1,1}^k > D_{1,2}^k > \cdots > D_{1,2m_k}^k
\]
(ordered from right-to-left on the x-axis).

3. Connect by lower semicircles centered midway between them, the following pairs of subintervals of \( C \):
\[
D_{1,1}^k \text{ to } D_{1,2}^k, D_{1,3}^k \text{ to } D_{1,4}^k, \ldots, D_{1,2m_k-1}^k \text{ to } D_{1,2m_k}^k.
\]

4. In \( D_2^k \) identify \( 2m_k \) equal subintervals of \( C \):
\[
D_{2,1}^k > D_{2,2}^k > \cdots > D_{2,2m_k}^k.
\]

5. Connect by lower semicircles centered midway between them, the following pairs of subintervals of \( C \):
\[
D_{2,2}^k \text{ to } D_{2,3}^k, D_{2,4}^k \text{ to } D_{2,5}^k, \ldots, D_{2,2m_k-2}^k \text{ to } D_{2,2m_k-1}^k.
\]

6. Let \( D_{1}^{k+1} = D_{2,1}^k \) and \( D_{2}^{k+1} = D_{2,2m_k}^k \).

The union of all upper and lower semicircles thus constructed is a continuum \( K_a \subset E^2 \). See Figure 7 for \( K_a \) based upon the sequence \( N_a = \{2,2,\cdots\} \).

We make the following observations about the way in which points of \( C \) are connected by arcs consisting piecewise of concentric upper and lower semicircles:

(a) \( D_1^k \) and \( D_2^k \) are connected by a Cantor set of parallel arcs consisting of upper semicircles, and concentric lower semicircles constructed in previous stages, in alternation.
(b) The partitions of $D_1^k$ and $D_2^k$ in steps (2) and (4) induce a partition of the Cantor set of arcs in (a) into $2m_k$ Cantor sets of (piecewise semicircular) arcs.

(c) $D_{1,m}^k$ is connected to $D_{2,2m_k-m+1}^k$ by the $m$th element of the induced partition in (b).

(d) The concentric lower semicircles constructed in steps (3) and (5) result in $D_{2,1}^k$ being connected to $D_{2,2m_k}^k$ by a Cantor set of piecewise semicircular arcs in which elements of the partition in (b) alternate with the sets of concentric lower semicircles constructed in (3) and (5).

(e) The order in which the $D_{1,m}^k$'s appear in the Cantor set of arcs connecting $D_{2,2m_k}^k$ to $D_{2,1}^k$ is:

$$D_{2,2m_k}^k, D_{1,1}^k, D_{1,2}^k, D_{2,2m_k-1}^k, D_{2,2m_k-2}^k, \ldots,$$

$$D_{2,2m_k-m}^k, D_{1,m+1}^k, D_{1,m+2}^k, D_{2,2m_k-m-1}^k, \ldots,$$

$$D_{2,2}^k, D_{1,2m_k-1}^k, D_{1,2m_k}^k, D_{2,1}^k.$$

Our conclusions in Section 3.2.2 below concerning the order in which certain points lie on $C_a$ follow from the above observations.

3.2.2 Composant $C_a$ of $K_a$. Composant $C_a$ of $K_a$ is a ray whose endpoint is $(0,0)$ and which includes $(1,0)$ and the endpoints of all deleted intervals, but does not include any other point of $C$. (The other points of $C$ are in the uncountably many other composants of $K_a$, each of which is a one-to-one continuous image of the reals.) We will coordinatize $C_a$ by $[0,\omega)$ by identifying $(0,0)$ with 0, midpoints of upper semicircles in $C_a$ with odd positive integers, and midpoints of lower semicircles in $C_a$ with even positive
Integers, in a manner to be described more fully below.

Let $\pi$ denote projection from $E^2$ into the $x$-axis. For $x,y \in E^2$, we say $x < y$ in projection order, meaning that $\pi(x) < \pi(y)$.

Note that all inner and outer semicircles are contained in $C_a$. We identify the midpoints of inner and outer semicircles as follows:

Let $a_{k,m}$ $(m = 1, \ldots , m_k)$ denote the midpoint of the outer semicircle in the collection of concentric lower semicircles joining $D_{1,2m-1}^k$ to $D_{1,2m}^k$. Call such points collectively $a$-points. Let $c_{k,m}$ $(m = 1, \ldots , m_k - 1)$ denote the midpoint of the inner semicircle in the concentric collection joining $D_{2,2m_k-2m}^k$ to $D_{2,2m_k-2m+1}^k$. Call such points collectively $c$-points. Note that the projection order of $a$- and $c$-points consists (after $a_{1,1} > \cdots > a_{1,m_1}$) of repeated sequences of the form:

$$
\cdots > a_{k,1} > a_{k,2} > \cdots > a_{k,m_k} > c_{k-1,m_k-1} > \\
c_{k-1,m_k-2} > \cdots > c_{k-1,1} > a_{k+1,1} > \cdots > \\
a_{k+1,m_k+1} > c_{k,m_k-1} > \cdots > c_{k,1} > a_{k+2,1} > \cdots 
$$

in which the $c$-points of stage $k-1$ are to the left (in the projection order) of the $a$-points of stage $k$, and therefore to the left of the $a$-points of stage $k-1$. However, in the order on $C_a$ we have the alternating sequence:

$$
\cdots < a_{k,1} < c_{k,1} < a_{k,2} < c_{k,2} < \cdots < \\
c_{k,m_k-1} < a_{k,m_k} < a_{k+1,1} < \cdots
$$

Let $b_{k,m}$ $(m = 1, \ldots , m_k - 1)$ denote the midpoint of the outer semicircle in the collection joining $D_{2,2m}^k$ to $D_{2,2m+1}^k$. 

Call such points collectively \( b\)-points. Note that the \( b\)-point \( b_{k,m} \) lies directly below the \( c\)-point \( c_{k,m_k-m} \). Let 
\[
d_{k,m} \ (m = 1, \ldots, m_k) \text{ denote the midpoint of the inner semi-circle in the collection joining } D_{k,2m_k-2m+1}^k \text{ to } D_{k,2m_k-2m+2}^k.
\]
Call such points collectively \( d\)-points. Note that \( d_{k,m} \) lies directly above \( a_{k,m_k-m+1} \). The projection order of \( b\)- and \( d\)-points consists (after \( d_{1,m_1} > \cdots > d_{1,1} \)) of repeated sequence of the form:
\[
\cdots > d_{k,m_k} > d_{k,m_k-1} > \cdots > d_{k,1} > b_{k-1,1} > \\
\cdots > b_{k-1,m_k-1} > d_{k+1,m_{k+1}} > \cdots > d_{k+1,1} > \\
b_{k,1} > \cdots > b_{k,m_k-1} > d_{k+2,m_{k+2}} > \cdots
\]
in which the \( d\)-points of stage \( k \) are to the right (in projection order) of the \( b\)-points of stage \( k-1 \), and therefore to the right of the \( b\)-points of stage \( k \).

Observe that the \( b\)- and \( d\)-points of stage \( k \) lie between \( a_{k+1,m_{k+1}} \) and \( a_{k+2,1} \) on \( C_a \). Consequently, in the order on \( C_a \), we have repeated sequences of the form:
\[
\cdots < a_{k+1,1} < c_{k+1,1} < a_{k+1,2} < \cdots < \\
a_{k+1,m_{k+1}-1} < c_{k+1,m_{k+1}-1} < a_{k+1,m_{k+1}} < d_{k,1} < \\
b_{k,1} < d_{k,2} < \cdots < d_{k,m_k-1} < b_{k,m_k-1} < d_{k,m_k} < \\
a_{k+2,1} < \cdots
\]

3.2.3 \textit{Pockets in } \( K_a \). Let \( A \) be an arc in \( E^2 \). We say that \( A \) is an \( \varepsilon\)-pocket iff there is an endcut \( B \subset E^2 - A \) to a point \( a \in A \) such that
(1) For each \( b \in B \), there is a crosscut \( Q \) to \( A \), transverse to \( B \) at \( b \), with \( \text{diam}(Q) < \varepsilon \).

(2) For each \( x \in A \), there is a \( y \in A \) such that \( x < a < y \) in some natural order on \( A \), \( d(x,y) < \varepsilon \), and there is a crosscut \( Q \) from \( x \) to \( y \) such that \( Q \) is transverse to \( B \) at some point \( b \in B \) and \( \text{diam}(Q) < \varepsilon \).

A crosscut \( Q \) is transverse to \( B \) at \( b \) iff \( Q \cap B = \{b\} \), and for a sufficiently small open disk neighborhood \( U \) of \( b \), each component of \( U - Q \) contains exactly one component of \( (U \cap B) - \{b\} \). By pocket we mean an \( \varepsilon \)-pocket for some \( \varepsilon \), usually small compared to \( \text{diam}(A) \).

Suppose that \( X \) is a connected subset of \( E^2 \) and \( A \subset X \) is an \( \varepsilon \)-pocket. Suppose there is a crosscut \( Q \) of \( X \) joining the endpoints of \( A \), with \( \text{diam}(Q) < \varepsilon \). Then \( Q \cup A \) is a simple closed curve. If \( X \subset \text{Cl} (\text{Ext}(Q \cup A)) \) for each sufficiently small crosscut \( Q \), we say that \( A \) is a pocket in \( X \).

If \( A \) is a pocket in \( X \), \( Q \) is a crosscut of \( X \) joining the endpoints of \( A \) so that \( X \subset \text{Cl} (\text{Ext}(Q \cup A)) \) and \( \text{diam}(Q) < \varepsilon \), and \( B \) is an endcut to \( A \) at point \( a \) satisfying conditions (1) and (2), then we call \( \text{Int}(Q \cup A) \) the inside of the pocket and \( \text{Ext}(Q \cup A) \) the outside of the pocket. We call the endpoints of the pocket the open end of the pocket, and we call the point \( a \in B \cap A \) the closed end of the pocket. We call the components of \( A - \{a\} \) the sides of the pocket. (Strictly speaking, the inside, outside, sides, and closed end of a pocket depend upon the choices made for \( Q \) and \( B \). In practice, this ambiguity will make little difference as \( \varepsilon \) is usually very small compared to \( \text{diam}(A) \).)
As we shall show in Section 3.3, the accessible points of $K_a$ are precisely the points in $C_a$. Thus pockets in $K_a$ are subarcs of ray $C_a$. We will show that the following four types of pockets occur in $C_a$:

* **aca-pockets**: a subarc of $C_a$ between two consecutive $a$-points of a given stage of construction, and the $c$-point between them as the closed end.

* **adb-pockets**: the subarc of $C_a$ between the last $a$-point of stage $k+1$ and the first $b$-point of stage $k$, and the $d$-point between them as closed end.

* **bdb-pockets**: a subarc of $C_a$ between two consecutive $b$-points of a given stage of construction, and the $d$-point between them as closed end.

* **bda-pockets**: the subarc of $C_a$ between the last $b$-point of stage $k$ and the first $a$-point of stage $k+2$, and the $d$-point between them as closed end.

Note that given $k$, there are $m_{k-1}$ aca-pockets between successive $a$-points of stage $k$; there are $m_{k-2}$ bdb-pockets between successive $b$-points of stage $k$; there is one adb-pocket, namely the subarc $[a_{k+1}, m_{k+1}, d_k, l, b_k, l]$; there is one bda-pocket, namely $[b_k, m_{k-1}, d_k, m_k, a_{k+2}, l]$. We show below that each of the above-mentioned pockets is a $2/2$ pocket in $K_a$.

Let $\mathcal{B}$ be the collection of midpoints of deleted intervals. Connect the points of $\mathcal{B}$ in pairs by a countable collection of upper semicircles centered at $(1/2, 0)$. At the $k^{th}$ stage in the construction of $K_a$ via the recursive procedure of Section 3.2.1, if $D_{i,m}^k$ is connected to $D_{i,m+1}^k$
by lower semicircles, then connect the points of ∂ that lie
between points of $D_{i,m}^k$ to those that lie between points of
$D_{i,m+1}^k$ by lower semicircles centered midway between $D_{i,m}^k$
and $D_{i,m+1}^k$. Let $r_{k,m}$ be the point in ∂ which lies midway
between $\pi(a_{k,m})$ and $\pi(a_{k,m+1})$ and let $s_{k,m} = \pi(c_{k,m})$, which
is in ∂, for $m = 1, \ldots, m_k - 1$. We obtain, for each $k$, and
each $m$, an arc $A_{k,m}^* \subset E^2 - K_a$ from $r_{k,m}$ to $s_{k,m}$ which is
the union of $2^{m_k-1}$ lower semicircles, alternating upper and
lower. Let $A_{k,m}$ be the endcut from $r_{k,m}$ to $c_{k,m}$ obtained
by extending $A_{k,m}^*$ by a vertical segment straight down to
c_{k,m}$. Then $A_{k,m}$ is clearly an endcut which shows that
subarc $[a_{k,m}, c_{k,m}, a_{k,m+1}]$ is a 2/2 $n_k$-pocket in $K_a$, for $A_{k,m}$
is parallel to the parallel arcs which form the sides of
the pocket (the sides are parallel except for small segments
containing $a_{k,m}, a_{k,m+1}$, and $c_{k,m}$). We call $A_{k,m}$ the
midline of the aca-pocket whose closed end is $c_{k,m}$.

In a similar fashion we construct the midline $B_{k,m}$ of
the pocket $[b_{k,m-1}, d_{k,m}, b_{k,m}]$ as an endcut from a point of
D midway between the projections of $b_{k,m-1}$ and $b_{k,m}$ to $d_{k,m}$
at the closed end of the pocket, for $m = 2, \ldots, m_k - 1$.
Midlines $B_{k,1}$ to $d_{k,1}$ at the closed end of pocket $[a_{k+1,m_k+1},$
d_{k,1}, b_{k,1}]$, and $B_{k,m_k}$ to $d_{k,m_k}$ at the closed end of pocket
$[b_{k,m_k-1}, d_{k,m_k}, a_{k+2,1}]$ are special cases only because the
endpoints of $B_{k,1}$ and $B_{k,m_k}$ that lie on the x-axis are the
points of ∂ midway between $D_{2,1}^k$ and $D_{2,2}$, and between
$D_{2,2m_k}$ and $D_{2,2m_k-1}$, respectively.
Observe that in projection order, the open end of an aca-pocket is closer to 0 than its closed end, while the open end of an adb-, bdb-, or bda-pocket is further from 0 than its closed end. Of course, both the open and closed ends of a pocket in $K_a$ are close to 0 compared to the diameter of the pocket.

Henceforth, for each $p$ in the collection of c- and d-points, let $M(p)$ denote the midline of the pocket whose closed end is the point $p$.

3.2.4 Cuts of $K_a$. A cut of $K_a$ is a Cantor set in $K_a$ which separates $K_a$. For example, each $D_{i,m}$ in our construction of $K_a$ is a cut of $K_a$. We identify below a convenient collection of cuts of $K_a$ which will be useful in showing the topological equivalence of $K_a$ and $K_b$, as well as showing the inequivalence of $K_a$ (based on sequence $N_a$) and $K_b$ (based on sequence $N_b$, distinct from $N_a$) in subsequent sections.

Let $L_{k,m}$ be the vertical segment joining $a_{k,m}$ to $d_{k,m'-m}^{k,m}$ for each $k$ and each $m = 1, \ldots, m_k$. Let $M_{k,m}$ be the vertical segment joining $c_{k,m}$ to $b_{k,m'-m}$ for each $k$ and each $m = 1, \ldots, m_k' - 1$. Then $L_{k,m}(M_{k,m})$ cuts $K_a$; that is each of $L_{k,m} \cap K_a$ and $M_{k,m} \cap K_a$ is a cut of $K_a$. Let $L_k = L_{k,1} \cup \cdots \cup L_{k,m_k}$, and let $M_k = M_{k,1} \cup \cdots \cup M_{k,m_k' - 1}$.

For distinct points $p$ and $q$ in the collection of all a-points, b-points, and (0,0) (we will refer to (0,0) as 0 hereafter), let $Q(p,q)$ denote a crosscut of small ($<2d(p,q)$) diameter from $p$ to $q$ lying below the x-axis. It is clear that such a crosscut exists, since a- and b-points are on outer semicircles. As before, for $p < q \in \mathbb{C}_a$, $[p,q]$
denotes the interval in $C_a$ between $p$ and $q$. Note that for fixed $p$ and $q$, the continuum in $K_a$ irreducible between the endpoints $p$ and $q$ of $Q(p,q)$ is $[p,q]$. Moreover, $Q(p,q) \cup [p,q]$ is a simple closed curve. For fixed $p$ and $q$, $Cl(\text{Int}(Q(p,q) \cup [p,q])) \cap K_a$ is independent of the choice of crosscut $Q(p,q)$, and is either $[p,q]$ (if neither $p$ nor $q$ is 0) or is $K_a$ (if either $p$ or $q$ is 0).

Observe that the two segments $L_{k,m}$ and $M_{k,m}$ together with the subarcs of $C_a$ between their endpoints, namely $[a_{k,m},c_{k,m}]$ and $[b_{k,m},d_{k,m},m+1]$, form a simple closed curve. A similar statement can be made for $M_{k,m}$ and $L_{k,m+1}$. Define the following closed subsets of $K_a$ which meet only in the indicated cuts (their boundaries relative to $K_a$), and which form a closed chain covering $K_a$:

$$[0,L_{k,1}] = Cl(\text{Int}(Q(0,a_{k+2,1}) \cup [0,a_{k,1}] \cup L_{1,1} \cup [d_{k,m},a_{k+2,1}])) \cap K_a,$$

$$[L_{k,m},M_{k,m}] = Cl(\text{Int}(L_{k,m} \cup [a_{k,m},c_{k,m}] \cup M_{k,m} \cup [b_{k,m},m,m+1])) \cap K_a,$$

$$[M_{k,m},L_{k,m+1}] = Cl(\text{Int}(M_{k,m} \cup [c_{k,m},a_{k,m+1}] \cup L_{k,m+1} \cup [b_{k,m},m,d_{k,m},m+1])) \cap K_a,$$

$$[L_{k,m+1},L_{k+1,m}] = Cl(\text{Int}(L_{k,m} \cup [a_{k,m},a_{k+1,1}] \cup Q(a_{k+1,1},a_{k+1,m+1},m+1) \cup [a_{k+1,1},d_{k,1}])) \cap K_a.$$ 

The notation is meant to suggest that the subset $[L_{k,m},M_{k,m}]$ lies between cuts $L_{k,m} \cap K_a$ and $M_{k,m} \cap K_a$, while the first
subset in the chain contains 0 and the last contains $L_{k+1}$. Actually, $[0, L_{k, 1}]$ also contain $\bigcup_{j=k+2}^{\infty} L_j \cap K_a$ and $\bigcup_{j=k+1}^{\infty} M_j \cap K_a$. The interiors (relative to $K_a$) of the above closed subsets constitute a partition of $K_a$ into $2m_k$ sets which meet only in their boundaries.

3.2.5 Topological equivalence of $K_a$ and $\hat{K}_a$. We use the closed covers of $K_a$ defined in Section 3.2.4 to define a sequence of retractions $r_k : K_a \to [0, a_k, 1] \subset C_a$, and we coordinate $C_a$ with the nonnegative reals so that $g_k$ is considered as a function with domain a subset of $C_a$ so that we satisfy the conditions

1. $r_k$ moves no point more than $\frac{2}{2^n_k}$, and
2. $r_k = g_k \circ r_{k+1}$.

It follows that $\lim_{k \to +} r_k = r : K_a \to \hat{K}_a$ is a homeomorphism.

Observe that $[0, a_k, 1] \subset [0, L_{k, 1}], \ldots, [a_k, m, c_{k, m}] \subset [L_k, m, M_k, m'], [c_{k, m}, a_{k, m+1}] \subset [M_k, m, L_k, m+1], \ldots, [a_k, m, a_{k+1}, 1] \subset [L_{k, m}, L_{k+1}]$. Since the $\frac{2}{2^n_k}$-neighborhood of each arc above contains the corresponding subset of $K_a$, it is easy to see that there is a retraction $r_{k+1} : K_a \to [0, a_{k+1}, 1]$ such that $r_{k+1}$ moves no point more than $\frac{2}{2^n_{k+1}}$, and such that $r_{k+1}([0, L_{k, 1}]) = [0, a_{k, 1}], \ldots, r_{k+1}([L_{k, m}, M_k, m]) = [a_{k, m}, c_{k, m}], r_{k+1}([M_k, m, L_{k, m+1}]) = [c_{k, m}, a_{k, m}], \ldots, r_{k+1}([L_{k, m}, L_{k+1}]) = [a_{k, m}, a_{k+1}, 1]$. However, in order to satisfy condition (2), we will need to define the retractions inductively, while coordinatizing $C_a$ so that condition (2) makes sense. It will turn out that with respect to $a$, $b$, $c$, and $d$-points, the coordinatization of $C_a$ is
(3) \( a_{k,m} = (2m-1)(2^{n_k}) \), for \( 1 \leq m \leq m_k \)

\( c_{k,m} = (2m)(2^{n_k}) \), for \( 1 \leq m \leq m_k-1 \)

(4) \( b_{k,m} = (2m_{k+1}-1)(2^{n_{k+1}}) + (2m)(2^{n_k}) \), for \( 1 \leq m \leq m_k-1 \)

\( d_{k,m} = (2m_{k+1}-1)(2^{n_{k+1}}) + (2m-1)(2^{n_k}) \), for \( 1 \leq m \leq m_k \)

We define the sequence \( \{r_k\}_{k=1}^{\infty} \) as follows: Recall that \( L_k = L_{k,1} \cup \cdots \cup L_{k,m_k} \) and \( M_k = M_{k,1} \cup \cdots \cup M_{k,m_k-1} \).

Let \( L = \bigcup_{k=1}^{\infty} L_k \) and let \( M = \bigcup_{k=1}^{\infty} M_k \). Then \( L \cup M \) cuts \( K_a \) into a Cantor set \( a \) of arcs, each arc, except for \([0,a_{1,1}]\), being composed of an upper semicircle and half of each of two lower semicircles. Let \( r_1: K_a \to [0,a_{1,1}] \) be a retraction so that \( r_1 \) is one-to-one on each arc in \( a \), \( r_1 \) moves no point more than \( 1/2 \), \( r_1^{-1}(0) = ((L-L_1) \cup M) \cap K_a \), and \( r_1^{-1}(a_{1,1}) = L_1 \cap K_a \). We may take \( r_1 \) to be radial projection from \((1/2,0)\) as center on upper semicircles, except near lower semicircles, and suitably modified so as to be one-to-one on the lower semicircles at each end of the upper semicircles. Coordinatize \([0,a_{1,1}]\) by \([0,2]\) so that the midpoint of the upper semicircle is identified with 1, \( a_{1,1} \) is identified with 2, and extend linearly in between (in terms of the length of \([0,a_{1,1}]\) as a curve in \( \mathbb{E}^2 \)).

In order to simplify notation, let \( L_{k,m}, M_{k,m}, L_k, M_k, L, \) and \( M \) denote the cuts of \( K_a \) for the remainder of this section, rather than the arcs which do the cutting. We now define \( r_2: K_a \to [0,a_{2,1}] \) and extend our coordinatization so that \([0,a_{2,1}]\) is identified with \([0,2^{n_2}]\), conditions
(1) and (2) are satisfied for \( k = 1,2 \), and (3) is satisfied for \( k = 1,2 \).

Observe that \( r_1^{-1}(2) \) is naturally partitioned into \( m_1 \) cuts, namely \( L_1,1,L_1,2,\ldots,L_1,m_1 \), and that \( r_1^{-1}(0) \) is partitioned into \( m + 1 \) cuts, namely \( (L-(L_1 \cup L_2)) \cup (M-M_1) \), \( M_1,1,M_1,2,\ldots,M_1,m_1-1,L_2 \). Moreover, for each \( t \in (0,2) \), \( r_1^{-1}(t) \) is naturally partitioned into \( 2m_1 \) cuts, namely \( [0,L_1,1] \cap r_1^{-1}(t),[L_1,1,M_1,1] \cap r_1^{-1}(t),\ldots,[L_1,m_1,L_2] \cap r_1^{-1}(t) \). For each \( t \in [0,2] \), \( r_1^{-1}(t) \) is a Cantor set, and the union over all \( t \) of such Cantor sets is \( K_a \).

Each such Cantor set is partitioned as noted above, and \( r_2 \) is defined as follows: For each \( t \in [0,2] \),

for \( x \in [0,L_1,1] \cap r_1^{-1}(t) \), \( r_2(x) = [0,a_1,1] \cap r_1^{-1}(t) \),

for \( x \in [L_1,1,M_1,1] \cap r_1^{-1}(t) \), \( r_2(x) = [a_1,1,c_1,1] \cap r_1^{-1}(t) \),

:\n
for \( x \in [L_1,m_1,M_1,m] \cap r_1^{-1}(t) \), \( r_2(x) = [a_1,m,c_1,m] \cap r_1^{-1}(t) \),

for \( x \in [M_1,m,L_1,m+1] \cap r_1^{-1}(t) \), \( r_2(x) = [c_1,m,a_1,m+1] \cap r_1^{-1}(t) \),

:\n
for \( x \in [L_1,m_1,L_2] \cap r_1^{-1}(t) \), \( r_2(x) = [a_1,m_1,a_2,1] \cap r_1^{-1}(t) \).

Since \( r_2 \) carries \([0,a_1,1] = [0,2] \) to itself by the identity, carries \([a_1,1,c_1,1] \) to \([0,a_1,1]\) one-to-one in reverse order,
carries \([c_1,1,a_{1,2}]\) to \([0,a_{1,1}]\) one-to-one in order, etc.,
the action of \(r_2\) on \([0,a_{2,1}]\) is similar to the projection
of the graph of \(g_2\) onto the second coordinate. We can
extend our coordinatization by identifying \([0,a_{2,1}]\) with
\([0,2^{n_2}]\) in such a way that \(r_1 = g_1 \circ r_2\), and (3) is satis-
fied for \(k = 1,2\). For example, if \(x \in [a_{1,1},c_{1,1}]\) and
\(r_1(x) = t\), then identify \(x\) with \((2)(2)-t\); if \(x \in [c_{1,1}a_{1,2}]\)
and \(r_1(x) = t\), then identify \(x\) with \((2)(2)+t\); if
\(x \in [a_{1,m_1},a_{2,1}]\) and \(r_1(x) = t\), then identify \(x\) with \(2^{n_2}-t\).

We now define \(r_3: K_a \leftrightarrow [0,a_{3,1}]\), extend our coordinati-

\(\text{zation so that } [0,a_{3,1}] \text{ is identified with } [0,2^{n_3}]\), satisfy
(1), (2), and (3) for \(k = 1,2,3\), and satisfy (4) for \(k = 1\).
Observe that \(r_2^{-1}(0) = (L-(L_1 \cup L_2)) \cup (M-M_1)\) and is parti-
tioned into \(m_2+1\) cuts, namely, \((L - \bigcup_{j=1}^{m_2}L_j) \cup ((M-(M_1 \cup M_2))\),
\(M_2,1, M_2,2, \ldots, M_2,2_{m_2-1}, L_3\). Also, \(r_2^{-1}(2^{n_2}) = L_2\), and is
partitioned into \(m_2\) cuts, namely, \(L_2,1, \ldots, L_2,m_2\). Moreover,
for each \(t \in (0,2^{n_2}) = (0,a_{2,1})\), \(r_2^{-1}(t)\) is naturally parti-
tioned into \(2m_2\) cuts, namely, \([0,L_2,1] \cap r_2^{-1}(t), [L_2,1,M_2,1] \cap r_2^{-1}(t), \ldots, [L_2,m_2,L_3] \cap r_2^{-1}(t)\). Therefore, we define \(r_3\)
in a fashion similar to the way we defined \(r_2\) above. For
each \(t \in [0,a_{2,1}] = [0,2^{n_2}]\),
for \(x \in [0,L_2,1] \cap r_2^{-1}(t), r_3(x) = [0,a_{2,1}] \cap r_2^{-1}(t),\)
for \(x \in [L_2,1,M_2,1] \cap r_2^{-1}(t), r_3(x) = [a_{2,1},c_{2,1}] \cap r_2^{-1}(t),\)
\(\vdots\)
for $x \in [L_2, m_2, L_3] \cap r_2^{-1}(t)$, $r_3(x) = [a_2, m_2, a_3, 1] \cap r_2^{-1}(t)$.

We extend our coordinatization by identifying $[0, a_3, 1]$ with $[0, 2^n 3]$ in such a way that (1), (2), and (3) are satisfied for $k = 3$, just as we did for $k = 2$. Moreover, since the $b$- and $d$-points in $[0, a_3, 1]$ all occur in the interval $[a_2, m_2, a_3, 1]$, and since each $d$-point lies above an $a$-point of stage $k = 1$, and each $b$-point lies below a $c$-point of stage $k = 1$, we have that (4) is satisfied for $k = 1$. Note that the $(2m_2 - 1)(2^n 2)$-term of the coordinatization of $d_1, m$ and $b_1, m$ is in fact the coordinate of $a_2, m_2$.

For each $k, k \geq 4$, we define $r_k$ in terms of $r_{k-1}$ and the previously defined cuts of $K_a$ just as we defined $r_3$ in terms of $r_2$ above.

We have therefore shown that for each sequence $N_a = \{a_1, a_2, \cdots\}$ of integers $a_k \geq 2$, there is an embedding $K_a$ of the Knaster U-continuum $K$ in $E^2$, with $K_a$ the natural embedding of an inverse limit system $\hat{K}_a$ topologically equivalent to $K$.

3.3 The prime end structure of $K_a$. Suppose that $K_a$ is the embedding of $K$ constructed in Section 3.2 on the basis of the sequence $N_a$ of integers each greater than 2. We use the notation of Section 3.2 below. Let $A$ denote the collection of $a$- and $b$-points of $C_a$, ordered as on $C_a$, and let $C$ denote the collection of $c$- and $d$-points of $C_a$, ordered as on $C_a$. For each $p \in A$, recall that $Q(0, p)$
denotes a small lower crosscut from \( 0 \) to \( p \). For each \( q \in C \), recall that \( M(q) \) denotes the midline of the pocket of which \( q \) is the closed end. Let \( M(0,q) \) denote the union of \( M(q) \) and a lower semicircle from \( 0 \) to the endpoint of \( M(q) \) on the \( x \)-axis. Then \( M(0,q) \) is a crosscut from \( 0 \) to \( q \).

3.3.1 Lemma. The set of accessible points of \( K_a \) is \( C_a \).

Proof. Every point of \([0,a_{1,1}] \cup A\) is clearly accessible. Every point of \( C_a - ([0,a_{1,1}] \cup A) \) lies in one of the pockets identified in Section 3.2.3. Hence, every point \( x \in C_a - ([0,a_{1,1}] \cup A) \) can be reached by a crosscut from \( 0 \) that follows \( M(0,x) \), if \( x \) is a c- or d-point, or that follows \( M(0,q) \), for some \( q \in C \), to a point \( y \in M(0,q) \), then follows half of a crosscut which is transverse to \( M(0,q) \) at \( y \) and has \( x \) as an endpoint. Hence \( C_a \) is accessible at every point.

Suppose \( x \) is an accessible point of \( K_a \). We claim \( x \in C_a \). Suppose not. For each \( p \in A \), \( Q(0,p) \cup [0,p] \) is a simple closed curve. Moreover, \( K_a \subset \text{Cl}(\text{Int}(Q(0,p) \cup [0,p])) \). Evidently, for \( p > q \in A \), \( \text{Cl}(\text{Int}(Q(0,p) \cup [0,p])) \subset \text{Cl}(\text{Int}(Q(0,q) \cup [0,q])) \), since we can chose that the crosscuts meet only in \( 0 \). In fact, \( K_a = \bigcap_{p \in A} \text{Cl}(\text{Int}(Q(0,p) \cup [0,p])) \). Recall that \( \text{diam}(Q(0,p)) < 2d(0,p) \).

Let \( R \) be an endcut to \( x \). Since \( x \notin C_a \), and in virtue of the towering of the simple closed curves above, with \( K_a \) being their intersection, either there is a first \( Q(0,p) \) that \( R \) doesn't meet, or \( R \) meets every \( Q(0,p) \). In the first case, clearly \( x \in [0,p] \). In the second case \( x = 0 \). Hence \( x \in C_a \), a contradiction.
3.3.2 Lemma. The prime ends of $E^2 - K_a$ are all trivial (of the first kind), except for that prime end $E$ defined by any chain of crosscuts converging to $0 \in C_a$.

Proof. If $Q$ is a crosscut of $E^2 - K_a$, then by Lemma 3.3.1, both endpoints of $Q$ lie in $C_a$. Let $J$ denote the arc of $C_a$ irreducible between the endpoints of $Q$. Then $Q \cup J$ is a simple closed curve, and either $K_a \subset \text{Cl}(\text{Int}(Q \cup J))$ or $\text{Cl}(\text{Int}(Q \cup J)) \cap K_a = J$.

Let $\{Q_i\}_{i=1}^\infty$ be a chain of crosscuts defining a prime end $F$ and let $J_i$ be the arc in $C_a$ irreducible between the endpoints of $Q_i$. If for some $j$, $\text{Cl}(\text{Int}(Q_j \cup J_j))$ fails to contain $K_a$, then for all $i \geq j$, $J_i \supset J_{i+1}$. Hence $F$ is trivial. So suppose $K_a \subset \text{Cl}(\text{Int}(Q_i \cup J_i))$ for all $i$. We claim that $\{Q_i\}_{i=1}^\infty$ converges to 0. Suppose not. Then the endpoints of the $Q_i$'s lie in $K_a - S(0, \varepsilon)$, where $S(0, \varepsilon)$ is an $\varepsilon$-ball about 0 and $\varepsilon$ is sufficiently small, for all but finitely many $Q_i$. There is a $j$ such that $\text{diam}(Q_i) < \varepsilon/2$ for all $i \geq j$. So we may assume that $Q_i \subset E^2 - \text{Cl}(S(0, \varepsilon/2))$, for all $i \geq j$. But then $Q_j$ misses a ray $R$ from 0 to $\infty$. Hence $K_a \not\subset \text{Cl}(\text{Int}(Q_j \cup J_j))$, a contradiction.

3.4 Proof of Theorem 1.2. Let $N_a = \{a_1, a_2, \ldots\}$ and $N_b = \{b_1, b_2, \ldots\}$ be sequences of integers such that for each $a_k \in N_a$ and each $b_j \in N_b$, $a_k \geq 2$ and $b_j \geq 2$. Let $K_a (K_b)$ be the embedding of $K$ following the procedures of Section 3.2 for $N_a (N_b)$. Let $C_a (C_b)$ denote the endpoint composant of $K_a (K_b)$ coordinatized by $[0, \infty)$ as previously described. It follows that the set of accessible points of $K_a (K_b)$ is $C_a (C_b)$ by Lemma 3.3.1. Since the endpoint
composant of $K$ is unique, $K_a$ and $K_b$ have the same set of accessible points. By Lemma 3.3.2, $K_a$ and $K_b$ have the same prime end structure.

To prove Theorem 1.2, it suffices to show that if $K_a$ and $K_b$ are equivalent embeddings, then $N_a$ and $N_b$ are not distinct sequences (Definition 2.2 and Lemma 2.3). Suppose that $h: \mathbb{E}^2 \to \mathbb{E}^2$ is a homeomorphism such that $h(K_a) = K_b$. Since our proof relies on constructions of crosscuts and midlines always in a compact neighborhood of $K_a$ or $K_b$, we lose no generality by assuming that $h$ and $h^{-1}$ are uniformly continuous.

Before proceeding further, we need to set up some notation. Let $n_1 = 1 = l$, let $n_k = 1 + \sum_{i=1}^{k-1} a_i$, and let $l_j = 1 + \sum_{i=1}^{j-1} b_i$. Let $m_k = 2^{a_k-1}$ and let $l_j = 2^{b_j-1}$. The a-, b-, c-, and d-points of $C_a$ and $C_b$, with their coordinatizations, are then designated as follows:

<table>
<thead>
<tr>
<th>$C_a$</th>
<th>$C_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a-points: $a_{k,m} = (2m-1)(2^{n_k})$</td>
<td>$a_{j,i} = (2i-1)(2^j)$</td>
</tr>
<tr>
<td>c-points: $c_{k,m} = (2m)(2^{n_k})$</td>
<td>$\gamma_{j,i} = (2i)(2^j)$</td>
</tr>
<tr>
<td>b-points: $b_{k,m} = (2m_k+1-1)(2^{n_{k+1}})$</td>
<td>$\beta_{j,i} = (2i_{j+1}+1)(2^{j+1})$</td>
</tr>
<tr>
<td></td>
<td>+ $(2m)(2^{n_k})$</td>
</tr>
<tr>
<td>d-points: $d_{k,m} = (2m_{k+1}+1)(2^{n_{k+1}})$</td>
<td>$\delta_{j,i} = (2i_{j+1}+1)(2^{j+1})$</td>
</tr>
<tr>
<td></td>
<td>+ $(2m-1)(2^{n_k})$</td>
</tr>
</tbody>
</table>

For a- and d-points of $C_a$, $1 \leq m \leq m_k$; for b- and c-points of $C_a$, $1 \leq m \leq m_k-1$. For a- and d-points of $C_b$, $1 \leq i \leq j$; for b- and c-points of $C_b$, $1 \leq i \leq i_j-1$. 

Let $A = \{a_1, l, a_1, 2, \ldots, a_1, m_1, a_2, 1, \ldots, a_2, m_2, b_1, l, \ldots, b_1, m_1-1, a_3, l, \ldots\}$ and $B = \{a_1, l, \ldots, a_1, i_1, a_2, l, \ldots, a_2, i_2, b_1, l, \ldots, b_1, i_1-1, a_3, l, \ldots\}$ be the sequences of $a$- and $b$-points of $C_a$ and $C_b$, respectively, ordered as they appear on $C_a$ and $C_b$. For $p \in A$ let $A(p)$ denote the infinite sequence that remains after removing the initial finite subsequence of points preceding $p$ in $A$. Similarly define $B(q)$ for $q \in B$.

The aca-, adb-, bdb-, and bda-pockets of $C_a$ and $C_b$ are constructed as described in Section 3.2.3. The critical step in our proof will be to show that aca-pockets of $C_a$ must be carried by $h$ to aca-pockets of $C_b$ after removing some initial finite subsequence of pockets from each, and that a similar correspondence exists between each of the remaining types of pockets. This will be shown to imply that $N_a$ and $N_b$ are not distinct.

3.4.1 Lemma. The homeomorphism $h$ has the following properties:

1. $h(C_a) = C_b$, with $h(0) = 0$.
2. $h$ is order-preserving from $C_a$ onto $C_b$.
3. There are points $p_0 \in A$ and $q_0 \in B$ such that we may assume $h(A(p_0)) = B(q_0)$; that is, $h|A(p_0) : A(p_0) \to B(q_0)$ is one-to-one and order-preserving onto $B(q_0)$.

Proof. Since $C_a$ ($C_b$) is the only composant of $K_a$ ($K_b$) which is a ray, properties (1) and (2) follow from the fact that $h|K_a$ is a homeomorphism onto $K_b$. Property (3),
however, follows from the fact that \( h \) is defined on \( E^2 \), as we shall show.

For each \( p \in A \), chose a short lower crosscut \( Q(0,p) \) (Section 3.2.4) so that for \( p \neq q \in A \), \( Q(0,p) \cap Q(0,q) = \{0\} \). We can do this because the reversed projection order on \( A \) duplicates the order of \( A \) (induced by \( C_a \)). Observe that 

\[
\text{diam}(Q(0,p)) \to 0.
\]

In a sufficiently small ball \( S(0,\varepsilon) \) about \( 0 \in C_b \), the components of \( C_b \cap S(0,\varepsilon) \) that contain an outer semicircle separate all other components of \( C_b \cap S(0,\varepsilon) \) from \( 0 \). Since \( \text{diam}(Q(0,p)) \to 0 \) implies that \( \text{diam}(h(Q(0,p))) \to 0 \), it follows that there is a \( p_1 \in A \) such that for all \( p \geq p_1 \in A \), \( h(p) \) lies on an outer semicircle of \( C_b \). (More precisely, \( h(p) \) lies on a component of \( C_b \cap S(0,\varepsilon) \) that contains an outer semicircle, for sufficiently small \( \varepsilon \).)

Suppose that infinitely often \( h \) carries a pair of consecutive points of \( A(p_1) \) to the same outer semicircle of \( C_b \). That is, suppose there is a sequence \( \{\{r_i,t_i\}\}_{i=1}^\infty \) of consecutive points of \( A(p_1) \) and a sequence \( \{C_i\}_{i=1}^\infty \) of outer semicircles of \( C_b \), such that \( h(r_i), h(t_i) \in C_i \). Then \( h([r_i,t_i]) \subset C_i \), since \( h \) is order-preserving onto \( C_b \).

But between each \( r_i \) and \( t_i \) on \( C_a \) there is a point \( s_i \) such that \( s_i + 1 \in C_a \). Hence \( h(s_i) + h(1) \in C_b \). However, \( C_i + 0 \in C_b \), and \( 0 \neq h(1) \). In view of this contradiction, we may suppose that there is a \( p_2 \in A \), \( p_2 \geq p_1 \), such that for all \( p \neq q \in A(p_2) \), \( h(p) \) and \( h(q) \) lie on different outer semicircles of \( C_b \).

Since each outer semicircle of \( C_b \) contains exactly one point of \( B \), we lose no generality by assuming that the
image of each point in \( A(p_2) \) is a point of \( B \). Hence we have that \( h|A(p_2): A(p_2) \rightarrow B \) is one-to-one and order-preserving with respect to the orders of \( C_a \) and \( C_b \), into \( B \). By repeating the preceding argument for \( h^{-1}: E^2 \rightarrow E^2 \) such that \( h^{-1}(K_b) = K_a \), we may assume that there is a \( q_1 \in B \) such that the image of each point of \( B(q_1) \) under \( h^{-1} \) is a point of \( A \). Hence we have that \( h^{-1}|B(q_1): B(q_1) \rightarrow A \) is one-to-one and order-preserving into \( A \).

Since \( A \) converges to \( 0 \in C_a \) and \( B \) converges to \( 0 \in C_b \), there is a \( q_2 \in B \), \( q_2 \geq q_1 \), such that \( h^{-1}(B(q_2)) \subset A(p_2) \).

Let \( p_3 = h^{-1}(q_2) \) in \( A(p_2) \). Then \( A(p_3) \subset A(p_2) \). If \( p \geq p_3 \in A \), then \( h(p) \geq h(p_3) = q_2 \). Hence \( h(p) \in B(q_2) \).

We claim that \( h|A(p_3): A(p_3) \rightarrow B(q_2) \) is onto. Let \( q \in B(q_2) \).

Then \( q \geq q_2 \), and since \( h^{-1}|B(q_2) \) is one-to-one and order-preserving into \( A(p_2) \), \( h^{-1}(q) \geq h^{-1}(q_2) = p_3 \). But then \( h^{-1}(q) \in A(p_3) \). Let \( p_0 = p_3 \) and \( q_0 = q_2 \), and property (3) is established.

Recall that on \( C_a (C_b) \), between each pair of consecutive points of \( A (B) \), there is a unique c- or d-point of \( C_a (C_b) \) which is the closed end of the pocket in \( C_a (C_b) \) of which the consecutive pair of points is the open end.

3.4.2 Lemma. There are points \( p_0 \in A \) and \( q_0 \in B \) such that if \( p_1 < p_2 \in A(p_0) \) are consecutive points and \( r \) is the c- or d-point that lies between them on \( C_a \), then we may assume that pocket \( [p_1, r, p_2] \subset C_a \) is carried by \( h \) onto a pocket \( [q_1, s, q_2] \subset C_b \), where we may assume that \( h(p_1) = q_1 \) and \( h(p_2) = q_2 \) are consecutive in \( B(q_0) \) and that \( s = h(r) \).
is the c- or d-point of $C_b$ that lies between $q_1$ and $q_2$.

**Proof.** By Lemma 3.4.1, we can find $p_0 \in A$ and $h(p_0) = q_0 \in B$ satisfying the lemma, except possibly for the condition that $h(r)$ be the c- or d-point of $C_b$ that lies between $q_1$ and $q_2$. However, since $h$ is order-preserving onto $C_b$, and since the c- and d-points of $C_a$ converge to 0, we may assume that $h(r)$ lies in some lower semicircle $C_1 \subset [q_1,q_2] \subset C_b$. Let $s$ be the c- or d-point of $C_b$ contained in $[q_1,q_2]$. There is only one such point, and it lies on the inner semicircle $C_2 \subset [q_1,q_2]$, of which there is only one contained in $[q_1,q_2]$. So, if we can show that $C_1$ and $C_2$ are actually the same lower semicircle, then we may assume that $h(r) = s$.

Suppose that $C_1$ and $C_2$ are distinct lower semicircles. Then $C_2$ contains the closed end of the pocket, and $C_1$ is on one side of the pocket $[q_1,s,q_2]$. Hence there are upper semicircles $C_3$ and $C_4$ in $[q_1,q_2]$ such that $C_3$ lies between $C_1$ and $C_2$ on $C_b$ and $C_4$ is parallel to $C_3$ but on the opposite side of the pocket. Let $x \in C_2$ and $y \in C_1$. We claim that any crosscut from $x$ to $y$ has $\text{diam}(Q) > 1/4$. We may assume that $p_0 \in A$ and $q_0 \in B$ have been chosen so that for all consecutive $q_1 < q_2 \in B(q_0)$, $[q_1,q_2]$ is an $\varepsilon$-pocket with $0 < \varepsilon < 1/16$ and $1/3-2\varepsilon > 1/4$, since $2/2^{1/2} > 0$. For each crosscut $R$ from $C_3$ to $C_4$ that is transverse to the midline $M(s)$ of pocket $[q_1,q_2]$ and has $\text{diam}(R) < \varepsilon$, $Q$ must meet $R$. Since $\text{diam}(C_3) > 1/3$, because $C_3$ is an upper semicircle, we have $\text{diam}(Q) > \text{diam}(C_3)-2\varepsilon > 1/3-2\varepsilon > 1/4$.

In the order on $C_b$ either $C_2 < C_1$ or $C_2 > C_1$. We assume that $C_2 < C_1$; the case for $C_2 > C_1$ is symmetric.
Since $h$ is uniformly continuous, chose $\delta > 0$ so that for all $x, y \in E^2$, if $d(x, y) < \delta$, then $d(h(x), h(y)) < \epsilon$. We may assume that $p_0 \in A$ and $h(p_0) = q_0 \in B$ have been chosen so that for all consecutive $p_1 < p_2 \in A(p_0)$, $[p_1, p_2]$ is a $\delta$-pocket, since $2/2^n \kappa + 0$. Let $M(r)$ be the midline of pocket $[p_1, r, p_2]$. Then for each $x \in [p_1, r)$, there is a $y \in (r, p_2]$ and a crosscut $Q$ from $x$ to $y$ transverse to $M(r)$ with $\text{diam}(Q) < \delta$. Thus $\text{diam}(h(Q)) < \epsilon$.

Since we have assumed that $C_2 < C_1$, we have $q_1 < s < h(r) < q_2$. Consequently, we have $p_1 < h^{-1}(s) < r < p_2$. There is an $x \in C_2$ so that $x < s$. Then $p_1 < h^{-1}(x) < h^{-1}(s) < r$. Since $[p_1, p_2]$ is a $\delta$-pocket, there is a $y \in (r, p_2]$ and a crosscut $Q$ from $h^{-1}(x)$ to $y$ such that $Q$ is transverse to $M(r)$ and $\text{diam}(Q) < \delta$. Then $h(Q)$ is a crosscut from $x$ to $h(y)$ and $\text{diam}(h(Q)) < \epsilon$.

Since $[q_1, q_2]$ is an $\epsilon$-pocket, and since $s < h(r) < q_2$, there is a $z \in [q_1, s)$ and a crosscut $R$ from $z$ to $h(r)$ such that $R$ is transverse to $M(s)$ and $\text{diam}(R) < \epsilon$. If it were the case that $z \in [x, s)$, then $z$ would be in $C_2$. Hence $R$ would be a crosscut from $z \in C_2$ to $h(r) \in C_1$, and so would have $\text{diam}(R) > 1/4$. Hence we must have $q_1 < z < x < s < h(r) < h(y) < q_2$.

We may assume that $R$ and $h(Q)$ are inside pocket $[q_1, q_2]$. Now $R \cup [z, h(r)]$ is a simple closed curve with $h(y)$ contained in $\text{Ext}(R \cup [z, h(r)])$. Moreover, $\text{Int}(R \cup [z, h(r)])$ is inside the pocket $[q_1, q_2]$ and its closure contains $x$. Hence crosscut $h(Q)$ from $x$ to $h(y)$ must meet crosscut $R$ from $z$ to $h(r)$. Thus $R \cup h(Q)$ contains a crosscut
T from x to h(r), and \( \text{diam}(T) \leq \text{diam}(h(Q)) + \text{diam}(R) < \varepsilon + \varepsilon < 1/8 \). However, T is a crosscut from \( x_1 \in C_2 \) to \( h(r_1) \in C_1 \), so \( \text{diam}(T) > 1/4 \). In view of this contradiction, it follows that \( C_1 \) and \( C_2 \) are the same lower semicircle.

It follows from Lemma 3.4.2 that we can find \( p_0 \in A \) and \( q_0 = h(p_0) \in B \), so that \( h|_{[p_0, \infty)} : [p_0, \infty) \to [q_0, \infty) \) carries pockets in \( [p_0, \infty) \subset C_a \) onto pockets in \( [q_0, \infty) \subset C_b \) in a one-to-one, order-preserving correspondence. However, given, say, an aca-pocket in \( [p_0, \infty) \), we have not yet determined what type of pocket in \( [q_0, \infty) \) its image under \( h \) must be. The next lemma establishes that \( h \) must preserve types of pockets from \( [p_0, \infty) \) onto \( [q_0, \infty) \).

3.4.3 Lemma. Let \( p_0 \in A \) and \( q_0 = h(p_0) \in B \) be chosen so as to satisfy Lemmas 3.4.1 and 3.4.2. Then \( h|_{[p_0, \infty)} : [p_0, \infty) \to [q_0, \infty) \) carries each aca-, adb-, bdb-, and bda-pocket in \( [p_0, \infty) \subset C_a \) onto a pocket in \( [q_0, \infty) \subset C_b \) of precisely the same type, in a one-to-one, order-preserving correspondence.

Proof. We argue in this first paragraph that the proof reduces to showing that the image under \( h \) of each aca-pocket of \( [p_0, \infty) \) is an aca-pocket of \( [q_0, \infty) \). Assume that is the case. Then by symmetry, the assumption also holds for aca-pockets of \( [q_0, \infty) \) and \( h^{-1}|_{[q_0, \infty)} : [q_0, \infty) \to [p_0, \infty) \). We may assume that \( p_0 \in A \) has been chosen so that \( p_0 = a_{k,1} \) for some \( k \). Consequently, by the order of \( a- \) and \( b- \) points on \( C_a \), the consecutive pockets of \( [p_0, \infty) \) appear in the following groupings: a finite number of aca-pockets, one
adb-pocket, a finite number of bdb-pockets, one bda-pocket, followed by the first aca-pocket of the next grouping. By our assumption, the image of each aca-pocket in \([p_0, \infty)\) is an aca-pocket in \([q_0, \infty)\). Consider the image of an adb-pocket of \([p_0, \infty)\). It cannot be an aca-pocket of \([q_0, \infty)\), because this would contradict our assumption regarding \(h^{-1}\). It could not be a bdb-pocket in \([q_0, \infty)\), since we know that its first endpoint must be an a-point of \(C_b\) because this endpoint is the last endpoint of the previous aca-pocket. Similarly, it cannot be a bda-pocket. So the image of an adb-pocket of \([p_0, \infty)\) must be an adb-pocket of \([q_0, \infty)\). By similar reasoning, the image of each bdb- and bda-pocket of \([p_0, \infty)\) must be, respectively, a bdb- and bda-pocket of \([q_0, \infty)\). Since the above argument holds for each grouping in turn on \([p_0, \infty)\), it follows that \(h|_{[p_0, \infty)}\) preserves types of pockets.

Let \(p_1 < p_2 \in A(p_0)\) be consecutive, so that \([p_1, r, p_2]\) is an aca-pocket in \([p_0, \infty)\). It suffices, by the above argument, to show that \([q_1, s, q_2]\) \(\subset [q_0, \infty)\) is also an aca-pocket, where \(h(p_1) = q_1\), \(h(p_2) = q_2\), and \(h(r) = s\), \(r\) being the c-point between \(p_1\) and \(p_2\) and \(s\) being the c-point between \(q_1\) and \(q_2\). (It follows from Lemma 3.4.2 that we may assume \(h(r) = s\) and that \(s\) is either a c-point or a d-point. We aim to show \(s\) must be a c-point.)

By way of contradiction, suppose that \([q_1, s, q_2]\) is not an aca-pocket of \([q_0, \infty)\). It certainly is a pocket, so we may assume, without loss of generality, that it is a bdb-pocket. The proof for the remaining two cases is almost exactly the same.
We may suppose that $p_0 \in A$ and $h(p_0) = q_0 \in B$ have been chosen to satisfy not only Lemmas 3.4.1 and 3.4.2, but also the conditions:

1. $1/16 > \eta > 0$ and $\eta > \varepsilon > 0$ are chosen, by uniform continuity of $h^{-1}$, so that for all $x, y \in E^2$ with $d(x, y) < \varepsilon$, $d(h^{-1}(x), h^{-1}(y)) < \eta$.

2. $\varepsilon > \delta > 0$ is chosen, by uniform continuity of $h$, so that for all $x, y \in E^2$ with $d(x, y) < \delta$, $d(h(x), h(y)) < \varepsilon$.

3. The outer and inner semicircles containing all $a$-, $b$-, $c$-, and $d$-points of $[p_0, \infty)$ are contained in $S(0, \delta)$.

4. The outer and inner semicircles containing all $a$-, $b$-, $c$-, and $d$-points of $[q_0, \infty)$ are contained in $S(0, \varepsilon)$.

As $r$ is a $c$-point of $C_a$, it lies on an inner semicircle $C(r)$ and there is a $b$-point $t$ of $C_a$ directly below $r$ and lying on an outer semicircle $C(t)$, parallel to $C(r)$. Since the closed end of an aca-pocket lies to the right of the open end in projection order, since $t$ is directly below $r$, and since reverse projection order duplicates the order on $C_a$ for points of $A$, we have $p_0 < p_1 < p_2 < t$ on $[p_0, \infty)$. Moreover, $C(r)$, $C(t)$, and all the lower semicircles parallel to them are contained in $S(0, \delta)$.

Let $L(r, t)$ be the vertical segment joining $r$ to $t$. Note that $L(r, t) \cap K_a$ is one of the cuts of $K_a$ previously defined. Also note that $L(r, t) \subset S(0, \delta) \subset S(0, \eta)$. Now $C(t)$, together with the short arcs in $S(0, \eta)$ of the two upper semicircles which meet $C(t)$ at its endpoints, locally separates $L(r, t) - \{t\}$ from 0; that is, $t$ is the only point of $L(r, t)$ that can be reached by a crosscut from 0 of
diameter less than $\eta$. (See Figure 8.) This holds because outer semicircles, together with short arcs of the upper semicircles which meet them, separate all lower semicircles parallel to them from 0 in a sufficiently small neighborhood of 0.

Since $h|_{[0,\infty)}$ is order-preserving, we have $q_0 \leq q_1 < q_2 < h(t)$ on $[q_0,\infty)$. We have assumed that $s = h(r)$ is the $d$-point at the closed end of the bdb-pocket $[q_1,s,q_2]$. Hence there is an $a$-point $u$ of $C_b$ directly below $s$. Let $C(s)$ denote the inner semicircle containing $s$ and $C(u)$ denote the outer semicircle containing $u$. Since the closed end of a bdb-pocket is to the right of the open end in projection order, since $u$ is directly below $s$, and since reverse projection order duplicates the order on $C_b$ for points of $B$, we have $0 < u < q_1 < q_2$ on $C_b$.

Let $C(u)^+$ denote $C(u)$ union the short arcs of the two upper semicircles that meet $C(u)$ contained in $S(0,\varepsilon)$. For any $x \in C(u)^+$, we have $0 < x < q_1 < q_2$. Moreover, $x$ can be reached by a crosscut from 0 contained in $S(0,\varepsilon)$.

Since $q_1 < q_2 < h(t)$, it follows that we have $h(t) \notin C(u)^+$. However, $h(t)$ lies on some outer semicircle $C(h(t))$ in $[q_0,\infty)$, since $h(t) \in B(q_0)$. Now $h(L(r,t))$ is an arc in $S(0,\varepsilon)$ from $s$ to $h(t)$, and $h(L(r,t))$ must cut $K_b$. Since $C(u)^+$ separates all other outer semicircles in $S(0,\varepsilon)$ from $s$, we have that $h(L(r,t)) \cap C(u)^+ \neq \emptyset$.

Let $x \in h(L(r,t)) \cap C(u)^+$. Obviously, $x \neq h(t)$. Let $Q$ be a crosscut from 0 to $x$, $Q \subset S(0,\varepsilon)$. Then $h^{-1}(Q)$ is a crosscut from 0 to $h^{-1}(x) \in L(r,t) - \{t\}$ with $h^{-1}(Q) \subset S(0,\eta)$. 
But this contradicts the fact that no point of \( L(r,t) - \{t\} \) can be reached in \( S(0,n) \) by a crosscut from 0. In view of this contradiction, \([q_1,s,q_2]\) cannot be a bdb-pocket.

3.4.4 Lemma. There are positive integers \( K \) and \( J \) such that the sequence \( N_a - \{a_1,a_2,\ldots,a_{K-1}\} \) is identical to the sequence \( N_b - \{b_1,b_2,\ldots,b_{J-1}\} \). Consequently, \( N_a \) and \( N_b \) are not distinct.

**Proof.** Let \( p_0 \in A \) and \( h(p_0) = q_0 \in B \) be chosen so as to satisfy Lemma 3.4.3. We may assume that for some \( K \), \( p_0 = a_{K,1} \in A \). Then \( A(p_0) = \{a_{K,1},a_{K,2},\ldots,a_{K,m_K},b_{K-1},\ldots\} \). Lemma 3.4.3 implies that

\[
h(a_{K,1}) = a_{J,I},
\]

for some \( J \) and some \( I, 1 \leq I \leq i_J \). This holds because \( h|_{[P_0,\infty)} \) preserves types of pockets. By similar reasoning, it follows that \( h(a_{K,2}) = a_{J,I+1}, \ldots, h(a_{K,m_K}) = a_{J,I+m_K-1}, h(b_{K-1,1}) = \beta_{J-1,1}, \) and so forth. In particular, it follows that \( h \) carries \( A(a_{K+1,1}) \) onto \( B(a_{J+1,1}) \) in a one-to-one order-preserving correspondence that, moreover, preserves types of points (a-points go to a-points, b-points go to b-points). It follows that \( m_{K+1} = i_{J+1} \) (from the a-points) and that \( m_K = i_J \) (from the b-points).

By the same reasoning applied to each successive grouping \( \{a_{K+r,1},\ldots,a_{K+r,m_{K+r}},b_{K+r-1,1},\ldots,b_{K+r-1,m_{K+r-1}},a_{K+r+1,1}\} \) carried by \( h \) onto the grouping \( \{a_{J+r,1},\ldots,a_{J+r,i_{J+r}},b_{J+r-1,1},\ldots,b_{J+r-1,i_{J+r-1}},a_{J+r+1,1}\} \) for each \( r \geq 2 \), it follows that for each \( r \geq 0 \), \( m_{K+r} = i_{J+r} \). Since
\[ m_{K+r} = 2^{a_{K+r}-1} \quad \text{and} \quad i_{J+r} = 2^{b_{J+r}-1}, \]

it follows that for all \( r \geq 0 \), \( a_{K+r} = b_{J+r} \) for \( a_{K+r} \in N_a \) and \( b_{J+r} \in N_b \). Therefore,
\[ N_a - \{a_1, a_2, \ldots, a_{K-1}\} \text{ is identical to } N_b - \{b_1, b_2, \ldots, b_{J-1}\}. \]

Hence \( N_a \) and \( N_b \) are not distinct.

In view of Lemma 3.4.4, if \( N_a \) and \( N_b \) are distinct sequences of integers greater than or equal to 2, then the corresponding embeddings \( K_a \) and \( K_b \) of \( K \) in \( E^2 \) are inequivalent. Since there are uncountably many such distinct sequences, Theorem 1.2 is established.

4. Uncountably Many Inequivalent Embeddings of Uncountably Many Knaster Continua

In [W] Will Watkins has shown that there are uncountably many nonhomeomorphic Knaster continua, answering a question posed by J. W. Rogers, Jr. in [R]. We have considered only the simplest of these continua, namely \( K^2 \), which is produced as the inverse limit of arcs under two-to-one bonding maps.

If we were to use instead six-to-one bonding maps, the resulting six-fold Knaster continuum \( K^6 \) would not be homeomorphic to \( K^2 \), though, like \( K^2 \), it would have only one endpoint.

For the sake of uniformity in describing a variety of Knaster continua, we alter our inverse limit notation slightly, so that the \( i^{\text{th}} \) bonding map will have domain the \( i^{\text{th}} \) space and range the \( (i-1)^{\text{th}} \) space. We will henceforth use \([0,1]\) as the \( 0^{\text{th}} \) space, where the bonding functions are indexed by \( i = 1, 2, \ldots \).

Let \( k, m, \) and \( n \) be positive integers such that \( n = km \).

Define a \( k\)-to-one rooftop function \( f_{n:m}^k : [0,n] \to [0,m] \) by
\[ f^n_m(rm) = \begin{cases} 
0, & \text{for even } r \\
m, & \text{for odd } r 
\end{cases} \]
and extend linearly between \( rm \) and \((r+1)m\), for \( 0 \leq r \leq k-1 \).

For positive integers \( n_1, n_2, \ldots, n_i \), let \( \prod_{j=1}^{i} n_j = n_1 n_2 \cdots n_i \) and let \( \prod_{j=1}^{0} n_j = 1 \).

4.1 Embeddings of the six-fold Knaster continuum.

Let the six-fold Knaster continuum \( K^6 \) be defined by the inverse limit system \( K^6 = \lim\{[0,6]^i, g_i\}_{i=1}^\infty \) where 
\[ g_i = f^6_{i-1} \]
is a six-to-one rooftop function. The standard embedding \( K_0^6 \) of \( K^6 \) in \( E^2 \) is illustrated in Figure 9. The Cantor set \( \mathcal{C}^6 \) used in the construction is not a middle third Cantor set, but rather, after the first stage, is a \( 5/11 \)-Cantor set. (In constructing a \( 5/11 \)-Cantor set, five of eleven equal subintervals are deleted from each subinterval of stage \( k \), so that there remain six disjoint intervals of stage \( k+1 \) in each interval of stage \( k \).)

For each sequence \( N_a = \{a_1, a_2, \ldots\} \) of positive integers, there is an inverse limit system 
\[ \hat{K}_a^6 = \lim\{[0,6^n], g_{n_i}\}_{i=1}^\infty \]
where \( n_0 = 0, n_i = \prod_{j=1}^{i} a_j \), and \( g_{n_i} = f^{6^n}_{n_i-1} \) is a 6\(^a_i\)-to-one rooftop function. Since a 6\(^a_i\)-to-one rooftop function is the composition of 6\(^a_i\) six-to-one rooftop functions, \( \hat{K}_a^6 \) is homeomorphic to \( K^6 \).

It can be shown, by techniques similar to those of Section 3, that there is a natural embedding \( K_a^6 \) of \( \hat{K}_a^6 \) in \( E^2 \). The recursive procedure for constructing the embedding is the same as that in Section 3.2.1, except that \( \mathcal{C}^6 \) replaces
and $6^{a_1}$ replaces $2^{a_1}$, while $6^{a_1/2}$ replaces $n_i$. Using the notion of pockets, it can be shown that if $K_a^6$ and $K_b^6$ are equivalent embeddings, then $N_a$ and $N_b$ are not distinct. Thus we obtain a result analogous to Theorem 1.2 for $K^6$.

All prime ends of $K_a^6$ are trivial, except for one of the second kind, and the set of accessible points is the unique endpoint composant.

4.2 Uncountably many nonhomeomorphic Knaster continua.

Following Watkins [W], we define an uncountable collection of nonhomeomorphic Knaster continua. Let $N = \{n_1, n_2, n_3, \ldots\}$ be a sequence of positive integers. Define the inverse limit system

\[ K^N = \lim_{\longrightarrow} \{[0,1]_{j=1}^\infty, g_{n_i} \}_{i=1}^\infty \]

where $g_{n_i} = f_{\prod_{j=1}^{n_i} n_i}$ is an $n_i$-to-one rooftop function.

If infinitely many of the $n_i$'s are even, then $K^N$ is a Knaster continuum with exactly one endpoint, called a U-type Knaster continuum. Continua $K_6^6$ and $K_2^6$ are both U-type Knaster continua, based, respectively, on the sequences \(\{2, 2, 2, \cdots\}\) and \(\{6, 6, 6, \cdots\}\). If only finitely many $n_i$'s are even, then $K^N$ is a continuum with exactly two endpoints, called an S-type Knaster continuum. The simplest S-type Knaster continuum is $K_3^3$ based on the sequence \(\{3, 3, 3, \cdots\}\). Obviously, no U-type Knaster continuum is homeomorphic to any S-type Knaster continuum.

Watkins constructs the uncountable collection of non-homeomorphic Knaster continua as follows: Let
P' = \{p_1', p_2', p_3', \ldots\} be a (finite or infinite) subset of the primes, ordered so that p_i < p_{i+1}. Form the infinite sequence

(2) P = \{p_1', p_1, p_2, p_2', p_3, p_3', p_4, \ldots\}

in which each p_i \in P' appears infinitely often. Re-index P without changing the order in which the primes appear so that P = \{p_1', p_2', p_3', \ldots\}. Let K^P be the inverse limit system defined by

(3) K^P = \lim_{i \to \infty} \left[0, 1 \right]_{p_i} \left[0, 1 \right]_{p_{i+1}} \ldots

as defined in (1) above, but with N = P.

Watkins shows that if P' and Q' are different subsets of primes (that is, there is a prime p \in P' - Q' or q \in Q' - P'), then, with P and Q formed from P' and Q' as in (2), the continua K^P and K^Q are not homeomorphic \([W, Theorem 3a]\). The conclusion that there are uncountably many nonhomeomorphic Knaster continua follows from the fact that there are uncountably many different subsets of the primes.

If the prime 2 should appear in P', then 2 will appear infinitely often in the sequence P. Hence K^P will be a U-type Knaster continuum. However, if 2 \notin P', then every element of P will be odd. Hence K^P will be an S-type Knaster continuum. Since there are uncountably many subsets of the primes which include (do not include) 2, there are uncountably many nonhomeomorphic U-type (S-type) Knaster continua.

The techniques of Section 3 can be adapted to S-type Knaster continua, but we restrict our attention to the U-type continua.
4.3 Uncountably many inequivalent embeddings of each of uncountably many $U$-type Knaster continua. Let $P' = \{p_1, p_2, p_3, \cdots\}$ be a subset of the primes with $p_1 = 2$.

We define an inverse limit system as follows: Let $m_i = \prod_{j=1}^{i} p_j$. Let $P^* = \{m_1, m_2, m_3, \cdots\}$. Let

$$K^{P^*} = \lim_{\rightarrow} \left\{ [0, \prod_{j=1}^{i} m_j], g_{m_i} \right\}_{i=1}^{\infty}$$

as defined in (1) above with $N = P^*$. Given $P'$, $K^P$ (see (3)) and $K^{P^*}$ are homeomorphic, since $g_{m_i} = g_{p_1} \circ g_{p_2} \circ \cdots \circ g_{p_i}$.

The advantage of $K^{P^*}$ is that each $m_i$ is even. This makes the standard embedding of $K^{P^*}$ easier to construct.

The standard embedding $K_0^{P^*}$ of $K^{P^*}$ in $\mathbb{R}^2$ is constructed much like the standard embedding of $K^6$. We start with a Cantor set $C^{P^*}$ in which the number of intervals deleted at each stage of construction depends upon $m_i$. Thus, we first delete a middle third interval in $I_0 = [0,1]$, leaving two intervals $I_{0,1}$ and $I_{0,2}$. We divide each $I_{0,j}$ into $2m_1 - 1$ equal intervals, and delete the "middle" $m_1 - 1$ of them, leaving $m_1$ disjoint intervals $I_{0,j,k}$ contained in each $I_{0,j}$.

We divide each $I_{0,j,k}$ into $2m_2 - 1$ equal intervals, delete the "middle" $m_2 - 1$ of them, leaving $m_2$ disjoint intervals in each $I_{0,j,k}$. We continue this process for each $m_i$, constructing the Cantor set

$$C^{P^*} = I_0 \cap (I_{0,1} \cup I_{0,2}) \cap \left( \bigcup_{j=1}^{m_1} I_{0,j,k} \right) \cap \left( \bigcup_{j=1}^{m_2} I_{0,j,k,l} \right) \cap \cdots.$$

The recursive procedure for constructing $K_0^{P^*}$, given $C^{P^*}$, is the same as the procedure in Section 3.2.1, except that $m_i$ replaces $2^{a_i}$ and $m_i/2$ replaces the $m_i$ of 3.2.1.
Let $N_a = \{a_1, a_2, a_3, \ldots \}$ be a sequence of positive integers. Let $n_0 = 0$, and let $n_i = \sum_{j=1}^{i} a_j$. Define the maps

$$g_{a_i} = g_{m_{n_i-1}+1} \circ \cdots \circ g_{m_{n_i-1}} \circ g_{m_{n_i}} : [0, \prod_{j=1}^{n_i} m_j] \to [0, \prod_{j=1}^{n_i-1} m_j]$$

Let $K_a^{*}$ be the inverse limit system

$$K_a^{*} = \lim\{[0, \prod_{j=1}^{n_i} m_j], g_{a_i}^{-1} \}_{i=1}^{\infty}$$

Since $g_{a_i}$ is the composition of the next $a_i$ unused $g_{m_j}$'s, $K_a^{*}$ is homeomorphic to $K_a^{P*}$.

The natural embedding $K_a^{P*}$ of $K_a^{*}$ in $E^2$ starts with the Cantor set $C^*$, constructed above for the standard embedding. However, the recursive procedure for constructing $K_a^{P*}$ replaces $2a_i$ in Section 3.2.1 with $r_i = \prod_{j=n_i-1}^{n_i} m_j$, and replaces the $m_i$ of Section 3.2.1 with $r_i/2$.

Techniques similar to those of Sections 3.2 through 3.4 can be used to prove an analog of Theorem 1.2 for each $K_a^{P*}$. That is, $K_a^{P*}$ has as its set of accessible points the unique ray-like composant of its endpoint; the prime end structure of $K_a^{P*}$ consists of trivial prime ends, except for one of the second kind; and, if $K_a^{b*}$ and $K_b^{P*}$ are equivalently embedded in $E^2$, then $N_a$ and $N_b$ are not distinct. Hence there are uncountably many inequivalent embeddings of $K_a^{P*}$. Moreover, there are uncountably many U-type Knaster continua for which the above statements are true.
4.4 Questions. We conclude with several questions concerning extensions of Theorem 1.2 to additional continua.

4.4.1 Question. Are there uncountably many inequivalent embeddings of the pseudo arc with the same prime end structure?

4.4.2 Question. Are there uncountably many inequivalent embeddings of every indecomposable chainable continuum (with the same prime end structure)?

4.4.3 Question. Are there uncountably many inequivalent embeddings of every (of some) indecomposable nonchainable continuum (with the same prime end structure)?

4.4.4 Question. Does Ingram's [I] atriodic nonchainable tree-like continuum have uncountably many inequivalent embeddings (with the same prime end structure)?

4.4.5 Question. Does the atriodic nonchaineable tree-like continuum of [M-1] have uncountably many inequivalent embeddings with the same prime end structure, specifically, all with a simple dense canal? Do all embeddings of that continuum have a simple dense canal?

We assume all the embeddings referred to above are into the plane. Answers to Questions 4 and 5, in particular may be of some relevance to the fixed point problem for nonseparating plane continua.
Figure 1. Standard embedding $K$ of $\sin \frac{1}{x}$ continuum.
Figure 2. Embedding $M_0$ of $\sin \frac{1}{x}$ continuum based on schema $\{S_o, S_o, S_o, \ldots \}$. 
Figure 3. Embedding of sin 1/x continuum based on schema \( \{S_2, S_1, \cdots \} \).
Figure 4. Embeddings of $\sin \frac{1}{x}$ continuum with same prime end structure as $K$.

Figure 5. Embeddings of $\sin \frac{1}{x}$ continuum with a prime end of the third kind.
Figure 6. Standard embedding $K_o$ of Knaster U-continuum (bucket handle).
Figure 7. Natural embedding $K_a$ of Knaster $U$-continuum $K$, defined by inverse limit $\tilde{K}_a$ based on $N_a = \{2, 2, \ldots\}$. 
Figure 8. Impossibility of $h$ carrying aca-pocket to bdb-pocket.
Figure 9. Standard embedding $K_6$ of six-fold Knaster (U-type) continuum $K_6$, defined by six-to-one bonding maps.
References


[Sm] M. Smith, *Uncountably many different embeddings of the pseudo-arc in the plane* (preprint).