WEAK CONFLUENCE AND MAPPINGS TO ONE-DIMENSIONAL POLYHEDRA

by

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1. Introduction

Throughout this paper the term mapping will mean a continuous function and a continuum will be a compact, connected metric space. Suppose X is a continuum, K is a subcontinuum of X, and f is mapping of a continuum onto X. The statement that f is weakly confluent with respect to K means some component of \( f^{-1}(K) \) is thrown by f onto K. The statement that f is weakly confluent means f is weakly confluent with respect to each subcontinuum of X.

Any mapping of a continuum onto a tree is weakly confluent with respect to each arc which does not contain a junction point in its interior. Many people, such as Read [7, Lemma p. 236], Ingram [3, Lemma 1], and Marsh [5, Lemma 4.7] have given a proof of some version of this, using the fact that the interior of such an arc separates the tree. Feurerbacher [2, Lemma 9] showed that if K is an arc in a circle S then any mapping of a continuum onto S must be weakly confluent with respect to K or \( S-K \).

In Theorem 4 of this paper we show that if \( K_1, \ldots, K_n \) is a collection of subcontinua of a one-dimensional polyhedron X whose interiors are mutually exclusive and contain no junction points, then the following are equivalent.

(1) Any mapping of a continuum onto X is weakly confluent with respect to one of \( K_1, \ldots, K_n \), and
(2) The union of the interiors of $K_1, \ldots, K_n$ separate $X$. In Theorem 5 we give conditions on the polyhedron which insure the separation in (2) above. We then use inverse limit representations of one-dimensional polyhedra to give conditions under which any mapping of a continuum onto a one-dimensional polyhedron $X$ must be weakly confluent with respect to one of a given collection of subcontinua of $X$.

The theorems in this paper can be used to show that certain one-dimensional continua are in $\text{Class}(W)$, where $\text{Class}(W)$ is the class of continua which are images of weakly confluent mappings only. We give an example of how these theorems may be used.

2. Weak Confluence and Separation of One-Dimensional Polyhedra

In this section we establish the main theorems of the paper.

Theorem 1. Suppose $X$ is a one-dimensional connected polyhedron and $K_1, K_2, \ldots, K_n$ are mutually exclusive non-degenerate subcontinua of $X$, no one of which contains a junction point or an endpoint of $X$. Then the following are equivalent:

(1) If $f$ is a mapping of a continuum onto $X$ then $f$ is weakly confluent with respect to one of $K_1, K_2, \ldots, K_n$.

and

(2) $X - \bigcup_{i=1}^{n} K_i$ is not connected.

Proof. (1) $\Rightarrow$ (2): Suppose $X - \bigcup_{i=1}^{n} K_i$ is connected. Let $A_1, A_2, \ldots, A_n$ be a mutually exclusive collection of arcs in $X$ such that $A_i \subseteq \text{Int } K_i$ for $i = 1, 2, \ldots, n$. Then
Let \( X - \bigcup_{i=1}^{n} A_i \) is connected and we denote by \( M \) the continuum 
\[ X - \bigcup_{i=1}^{n} A_i. \]

We define a mapping \( f \) of \( M \) onto \( X \) which is not weakly confluent with respect to any \( K_i \). For \( i = 1, 2, \ldots, n \), \( \overline{K_i - A_i} \) is the union of two mutually exclusive arcs \( \alpha_i \) and \( \beta_i \) each of which has one endpoint which is an endpoint of \( K_i \) and one endpoint which is an endpoint of \( A_i \). We define \( f|\alpha_i \) and \( f|\beta_i \) so that \( f|\alpha_i \) is a homeomorphism which maps \( \alpha_i \) onto \( \alpha_i \cup A_i \) and \( f|\beta_i \) is a homeomorphism which maps \( \beta_i \) onto \( \beta_i \cup A_i \), and so that the endpoint of \( K_i \) belonging to \( \alpha_i \) is a fixed point of \( f|\alpha_i \) and the endpoint of \( K_i \) belonging to \( \beta_i \) is a fixed point of \( f|\beta_i \). We define \( f|(X - \bigcup_{i=1}^{n} K_i) \) to be the identity mapping on \( X - \bigcup_{i=1}^{n} K_i \).

For \( i = 1, 2, \ldots, n \), \( f^{-1}(K_i) \) has two components, \( \alpha_i \) and \( \beta_i \). Neither \( f(\alpha_i) \) nor \( f(\beta_i) \) is \( K_i \), hence \( f \) is not weakly confluent with respect to \( K_i \).

(2) \( \Rightarrow \) (1): Suppose \( X - (\bigcup_{i=1}^{n} K_i) \) is not connected. Let \( f \) be a mapping of a continuum \( M \) onto \( X \).

**Case 1.** \( X - K_1 \) is not connected. Let \( A \) be an arc in \( X \) containing no junction point or endpoint of \( X \) such that \( K_1 \subseteq \text{Int} \, A \). Then \( X - A \) is not connected and has only two components, \( C_1 \) and \( C_2 \). Let \( a_1 \in \overline{C_1 \cap A} \) and \( a_2 \in \overline{C_2 \cap A} \). Let \( g \) be the mapping of \( X \) onto \( A \) defined by

\[
g(x) = \begin{cases} 
    a_1 & \text{if } x \in \overline{C_1} \\
    a_2 & \text{if } x \in \overline{C_2} \\
    x & \text{if } x \in A.
\end{cases}
\]

The composition \( g \circ f \) is a mapping of \( M \) onto the arc \( A \) and so by [7, Lemma p. 236] \( g \circ f \) is weakly confluent. Thus,
there is a subcontinuum $H$ of $M$ such that $gof(H) = K_1$. Since $f(H)$ is a continuum in $X$ which is thrown by $g$ onto $K$, $g|A$ is a homeomorphism, and $g^{-1}(K_1) = K_1$, then $f(H) = K_1$. Therefore, $f$ is weakly confluent with respect to $K_1$.

Case 2. $X - K_1$ is connected. Let $m$ be a positive integer less than $n$ such that $X - \bigcup_{i=1}^{m} K_i$ is connected and $X - \bigcup_{i=1}^{m+1} K_i$ is not connected. Let $A_1, A_2, \ldots, A_{m+1}$ be mutually exclusive arcs in $X$, no one of which contains a junction point or an endpoint of $X$, such that $K_i \subseteq \text{Int } A_i$ for $i = 1, \ldots, m+1$. Then $X - \bigcup_{i=1}^{m} A_i$ is connected and $X - \bigcup_{i=1}^{m+1} A_i$ is not connected. Since $X - \bigcup_{i=1}^{m+1} A_i = (X - \bigcup_{i=1}^{m} A_i) - A_{m+1}$, then $X - \bigcup_{i=1}^{m+1} A_i$ has only two components, $C_1$ and $C_2$.

Let $a_1 \in A_1 \cap \overline{C_1}$ and $a_2 \in A_2 \cap \overline{C_2}$. Let $g$ be the mapping of $X$ onto $A_1$ defined by

$$g(x) = \begin{cases} x & \text{if } x \in A_1 \\ a_1 & \text{if } x \in \overline{C_1} \\ a_2 & \text{if } x \in \overline{C_2} \end{cases}$$

and for $i = 2, \ldots, m+1$, we define $g|A_i$ to be a homeomorphism which throws $A_1$ onto $A_i$ in such a way that $g(K_1) = K_1'$, $g(A_1 \cap \overline{C_1}) = \{a_1\}$, and $g(A_1 \cap \overline{C_2}) = \{a_2\}$.

The composition $gof$ is a mapping of $M$ onto the arc $A$, and so by [7, Lemma p. 236] $gof$ is weakly confluent. Thus, there is a subcontinuum $H$ of $M$ such that $gof(H) = K_1$. Since $f(H)$ is a continuum in $X$ which is thrown by $g$ onto $K_1$, $g^{-1}(K_1) = \bigcup_{i=1}^{m+1} K_i$, and $g|A_i$ is a homeomorphism for $i = 1, \ldots, m+1$, then $f(H)$ is one of $K_1, \ldots, K_{m+1}$. Therefore, $f$ is weakly confluent with respect to one of $K_1, \ldots, K_{m+1}$. 


The next theorem gives conditions which insure the separation in (2) of Theorem 1.

To each metric space $X$ there corresponds a non-negative integer $b_1(X)$ (see [4, p. 409]). If $X$ is a polyhedron, $b_1(X)$ is the one-dimensional Betti number of $X$.

**Theorem 2.** Suppose $X$ is a one-dimensional connected polyhedron and $n$ is non-negative integer such that $b_1(X) = n$, and $K_1, K_2, \ldots, K_{n+1}$ are mutually exclusive subcontinua of $X$, no one of which contains a junction point or an endpoint of $X$. Then $X - \bigcup_{i=1}^{n+1} K_i$ is not connected.

**Proof.** Suppose $X - \bigcup_{i=1}^{n+1} K_i$ is connected. Let $A_1, A_2, \ldots, A_{n+1}$ be mutually exclusive arcs in $X$, no one of which contains a junction point or an endpoint of $X$, such that $K_i \subseteq \text{Int } A_i$, for $i = 1, 2, \ldots, n+1$. Then $X - \bigcup_{i=1}^{n+1} A_i$ is connected.

Let $a$ be an endpoint of $A_1$ and $g$ be the mapping of $X$ into $X$ defined by

$$g(x) = \begin{cases} a & \text{if } x \in X - \bigcup_{i=1}^{n+1} A_i \\ x & \text{otherwise} \end{cases}$$

Since $g$ is a monotone mapping, it follows from [4, Theorem 4, p. 433] that $b_1(X) \geq b_1(g(X))$. But $g[X]$ has only $n+1$ simple closed curves and one junction point; hence, $b_1(g(X)) = n + 1$. This yields a contradiction.

The next theorem follows from Theorem 1 and 2.

**Theorem 3.** Suppose $X$ is a one-dimensional connected polyhedron, $n$ is a non-negative integer such that $b_1(X) = n$,
and $K_1, K_2, \ldots, K_{n+1}$ are mutually exclusive non-degenerate subcontinua of $X$, no one of which contains a junction point or an endpoint of $X$. If $f$ is a mapping of a continuum onto $X$ then $f$ is weakly confluent with respect to one of $K_1, \ldots, K_{n+1}$.

In Theorems 4, 5, and 6 we relax the conditions regarding junction points imposed on the collections of subcontinua in the hypotheses of Theorems 1, 2 and 3.

Theorem 4. Suppose $X$ is a one-dimensional polyhedron, $K_1, \ldots, K_n$ are non-degenerate subcontinua of $X$ whose interiors are mutually exclusive, and no one of $K_1, K_2, \ldots, K_n$ contains a junction point of $X$ in its interior. Then the following are equivalent.

(1) If $f$ is a mapping of a continuum onto $X$ then $f$ is weakly confluent with respect to one of $K_1, K_2, \ldots, K_n$,

and

(2) $X - \bigcup_{i=1}^n \operatorname{Int} K_i$ is not connected.

Proof. (1) $\Rightarrow$ (2): Suppose $X - \bigcup_{i=1}^n \operatorname{Int} K_i$ is not connected. Let $f$ be a mapping of a continuum $M$ onto $X$. For each $i = 1, 2, \ldots, n$, let $A^1_i, A^2_i, \ldots$ be a sequence of arcs such that $A^1_j \subseteq \operatorname{Int} K_i$, $A^1_j \subseteq A^1_{j+1}$, and $\lim_{j \to \infty} A^1_j = K_i$.

Then for each positive integer $j$, $A^1_j, A^2_j, \ldots, A^n_j$ are mutually exclusive subcontinua of $X$, no one of which contains a junction point or an endpoint of $X$. Since $X - \bigcup_{i=1}^n A^i_j$ is not connected, then $X - \bigcup_{i=1}^n A^i_j$ is not connected. Then, by Theorem 1, $f$ is weakly confluent with respect to one of $A^1_j, A^2_j, \ldots, A^n_j$. 

There exists a positive integer $i$ such that $f$ is weakly confluent with respect to infinitely many of $A_1^i, A_2^i, \ldots$. Thus, there is a sequence $L_1, L_2, \ldots$ of subcontinua of $M$ such that $f(L_1), f(L_2), \ldots$ is a subsequence of $A_1^i, A_2^i, \ldots$. We choose a subsequence $L_{m_1}, L_{m_2}, \ldots$ of $L_1, L_2, \ldots$ which converges to a subcontinuum $L$ of $M$. Then $f(L) = \lim_{j \to \infty} f(L_{m_j}) = \lim_{j \to \infty} A_j^i = K_i$. Therefore, $f$ is weakly confluent with respect to $K_i$.

$(2) \Rightarrow (1)$: Suppose that $X - \bigcup_{i=1}^n \text{Int } K_i$ is connected. Let $A_1, A_2, \ldots, A_n$ be $n$ arcs in $X$ such that $A_i \subseteq \text{Int } K_i$ for $i = 1, 2, \ldots, n$. Then $X - \bigcup_{i=1}^n A_i$ is connected, and so, by Theorem 1, there exists a continuum $M$ and a mapping $f$ of $M$ onto $X$ such that $f$ is not weakly confluent with respect to $A_i$, for each $i = 1, 2, \ldots, n$. Since, for each $i$, $K_i - A_i$ is not connected, it follows from Theorem 1 that $f$ is not weakly confluent with respect to $K_i$, for each $i = 1, 2, \ldots, n$.

Theorem 5. Suppose $X$ is a one-dimensional connected polyhedron and $n$ is a non-negative integer such that $b_1(X) = n$ and $K_1, K_2, \ldots, K_{n+1}$ are subcontinua of $X$ whose interiors are mutually exclusive, and no one of $K_1, K_2, \ldots, K_{n+1}$ contains a junction point of $X$ in its interior. Then $X - \bigcup_{i=1}^{n+1} \text{Int } K_i$ is not connected.

Proof. Suppose $X - \bigcup_{i=1}^{n+1} \text{Int } K_i$ is connected. Let $A_1, A_2, \ldots, A_{n+1}$ be subcontinua of $X$ such that $A_i \subseteq \text{Int } K_i$, for $i = 1, 2, \ldots, n+1$. Then $A_1, A_2, \ldots, A_{n+1}$ are mutually exclusive subcontinua of $X$, no one of which contains a junction point or an endpoint of $X$, and $X - \bigcup_{i=1}^{n+1} A_i$ is connected. This contradicts Theorem 2.
The next theorem follows from Theorems 4 and 5.

**Theorem 6.** Suppose that \( X \) is a one-dimensional connected polyhedron, \( n \) is a non-negative integer such that \( b_1(X) = n \), and \( K_1, K_2, \ldots, K_{n+1} \) are non-degenerate subcontinua of \( X \) whose interiors are mutually exclusive, and no one of \( K_1, K_2, \ldots, K_{n+1} \) contains a junction point of \( X \) in its interior. If \( f \) is a mapping of a continuum onto \( X \) then \( f \) is weakly confluent with respect to one of \( K_1, K_2, \ldots, K_{n+1} \).

Theorem 5 shows that in a one-dimensional connected polyhedron \( X \), any collection of at least \( b_1(X) + 1 \) subcontinua of \( X \) which satisfy certain conditions must separate \( X \). The following theorem shows that it is necessary to require this many subcontinua to assure separation.

**Theorem 7.** Suppose \( X \) is a one-dimensional connected polyhedron and \( n \) is a positive integer such that \( b_1(X) = n \). Then there exist \( n \) mutually exclusive subcontinua \( K_1, K_2, \ldots, K_n \) of \( X \), no one of which contains a junction point or an endpoint of \( X \), such that \( X - \bigcup_{i=1}^{n} K_i \) is connected.

**Proof.** Since \( b_1(X) > 1 \) then \( X \) contains a simple closed curve. Let \( K_1 \) be an arc in this simple closed curve which contains no junction point of \( X \). Then \( X - K_1 \) is connected, so by the Euler-Poincaré formula [6, Theorem 9, p. 32]

\[
b_1(X - K_1) = b_1(X) - 1.
\]

One can see this by noting that \( X - K_1 \) has one more 1-simplex and two more 0-simplexes than \( X \).)

We define, inductively, arcs \( K_2, \ldots, K_n \) in \( X \) such that for \( j = 2, \ldots, n \) \( K_j \) is in a simple closed curve in \( X - \bigcup_{i=1}^{j-1} K_i \).
$K_j$ contains no junction point of $X$, and $X - \bigcup_{i=1}^{j-1}K_i$ is connected. By the Euler-Poincaré formula,

$$b_1(X - \bigcup_{i=1}^{j-1}K_i) = b_1(X - \bigcup_{i=1}^{j-1}K_i) - 1 = b_1(X) - \sum_{i=1}^{j}i.$$  

Therefore, $X - \bigcup_{i=1}^{n}K_i$ is connected.

In the next theorem, we show that the conditions regarding junction points imposed on the collection of subcontinua in Theorem 4 may not be weakened.

**Theorem 8.** Suppose $X$ is a one-dimensional connected polyhedron and $K_1, K_2, \ldots, K_n$ are mutually exclusive proper subcontinua of $X$ such that each of $K_1, K_2, \ldots, K_n$ contains a junction point of $X$ in its interior. Then there exists a continuum $M$ and a mapping $f$ of $M$ onto $X$ such that $f$ is not weakly confluent with respect to $K_i$ for each $i = 1, 2, \ldots, n$.

**Proof.** We show there is a continuum $M$ and a mapping $f$ of $M$ onto $X$ which is not weakly confluent with respect to $K_1$. There is a point $x$ in $X - \bigcup_{i=1}^{n}K_i$ and an arc $a = [x,a]$ such that $a \notin K_1$, $[x,a] \cap (\bigcup_{i=1}^{n}K_i) = \emptyset$, and $[x,a]$ contains no junction point of $X$. Let $J$ be a junction point of $X$ in $\text{Int } K_1$ and let $\beta = [a,J]$ be an arc in $K_1$ joining $a$ and $J$.

**Case 1.** There is an arc $[t,J]$ such that $[t,J] \cap (\bigcup_{i=1}^{n}K_i) = \emptyset$ and $[t,J]$ contains no junction point of $X$. Let $[k,J]$ be an arc in $\beta$ such that $[k,J]$ contains no junction point of $X$. Let $M$ be the union of the following three subsets of $X \times [0,1]$:
Let $f$ be the projection mapping of the continuum $M$ onto $X$.

We show that $f$ is not weakly confluent with respect to $K$. Suppose there is a subcontinuum $H$ of $M$ such that $f(H) = K$. Since $J$ is in the interior of $K$, there is a point $y$ in $K$ such that $y \notin \alpha \cup \beta \cup [k,J]$. Now, $f^{-1}(y) = \{(y,0)\}$, $f^{-1}([k,J]) = (k,J) \times \{1\}$, and $f|\{(k,J) \times \{1\}\}$ is one to one. Thus, $H$ must contain the point $(y,0)$ and a point of $(k,J) \times \{1\}$. But, any subcontinuum of $M$ which contains $(y,0)$ and a point of $(k,J) \times \{1\}$ must intersect one of $\{t\} \times [0,1]$ and $\{x\} \times [0,1]$. Therefore, the image of such a continuum under $f$ must contain a point not in $K$, and so $f$ is not weakly confluent with respect to $K$.

**Case 2.** Case 1 does not hold. Then there exist two arcs $[r,J]$ and $[s,J]$ such that $[r,J] \cup [s,J] \subseteq K \setminus \beta$ and neither $[r,J]$ nor $[s,J]$ contains a junction point of $X$.

We will resolve this case in two parts. First, suppose that $X \setminus (r,J)$ is not connected. Then $X \setminus (r,J)$ has only two components, one containing $r$ and the other containing $J$, $s$, and $x$. Let $M$ be the union of the following three subsets of $X \times [0,1]$:

$$[X \setminus (r,J)] \times \{0\},$$

$$\alpha \cup \beta \cup (r,J) \times \{1\},$$

and

$$\{x,r\} \times [0,1].$$

Let $f$ be the projection mapping of $M$ onto $X$. 
We show that $f$ is not weakly confluent with respect to $K_1$. If $H$ is any subset of $M$ such that $K_1 \subseteq f(H)$ then $H$ must contain the point $(s,0)$ and a point of $(r,J) \times \{1\}$. But, any continuum in $M$ containing two such points must intersect $\{x\} \times [0,1]$, and hence $f(H)$ contains points not in $K_1$. Thus, $f$ is not weakly confluent with respect to $K_1$.

On the other hand, suppose that $X - (r,J)$ is connected. Let $M$ be the union of the following three subsets of $X \times [0,1]$:

- $[X - (r,J)] \times \{0\}$,
- $(a \cup b \cup [r,J]) \times \{1\}$, and
- $\{x\} \times [0,1]$.

Let $f$ be the projection mapping of the continuum $M$ onto $X$.

We show that $f$ is not weakly confluent with respect to $K_1$. If $H$ is any subcontinuum of $M$ such that $K_1 \subseteq f(H)$ then $H$ must contain $(r,J) \times \{1\}$ and the point $(s,0)$. But, any continuum in $M$ containing such points must intersect $\{x\} \times [0,1]$ and hence $f(H)$ contains points not in $K_1$. Thus, $f$ is not weakly confluent with respect to $K_1$. This concludes Case 2.

In each case, $M$ was constructed by removing an arc from $X$ and building a bridge over it in $X \times [0,1]$. In doing this we were careful to stay away from $\bigcup_{i=2}^{n} K_i$. This construction can be repeated for each of $K_2, \ldots, K_n$, resulting in a continuum $M'$ in $X \times [0,1]$ such that the projection mapping $f'$ of $M'$ onto $X$ is not weakly confluent with respect to $K_1$, for each $i = 1, 2, \ldots, n$. 
Remark. It is interesting to note that with $M'$ so constructed, one can see that an arc can be mapped onto $M'$ in such a way that the composition of this mapping with $f'$ is not weakly confluent with respect to any $K_i$. Thus, we may assume the continuum $M$ in the statement of Theorem 8 is an arc.

3. Inverse Limits

In this section we use inverse limit representations of one-dimensional polyhedra to describe conditions under which any mapping of a continuum onto a one-dimensional polyhedron $X$ must be weakly confluent with respect to one member of a given collection of subcontinua of $X$. These results can be used to show that certain one-dimensional continua are in Class(W).

Suppose $X_1, X_2, \cdots$ is a sequence of compact metric spaces each having diameter less than a fixed positive number $c$, and suppose $f_1, f_2, \cdots$ is a sequence of mappings such that $f_i$ maps $X_{i+1}$ onto $X_i$ for $i = 1, 2, \cdots$. The inverse limit of the inverse limit sequence $\{X_i, f_i\}$ is the subset of the product $\prod X_i$ to which $(x_1, x_2, \cdots)$ belongs if and only if $f_n(x_{n+1}) = x_n$ for $n = 1, 2, \cdots$. We consider $\prod X_i$ metrized by

$$d(x, y) = \prod_{i > 0} 2^{-i} d_i(x_i, y_i)$$

where $d_i$ denotes the metric on $X_i$. For each $i = 1, 2, \cdots$, $\pi_i$ will denote the projection mapping of the inverse limit onto $X_i$. 
The following lemma was essentially proved by Read [7, Theorem 4] although not stated in this way. A proof is included here only for the sake of completeness.

**Lemma 1.** Suppose $X$ is the inverse limit of the inverse limit sequence $\{X_i, f_i\}$ with each $X_i$ a continuum, $K$ is a subcontinuum of $X$, and $g$ is a mapping of a continuum onto $X$. If $\pi_i \circ g$ is weakly confluent with respect to $\pi_i(K)$ for infinitely many integers $i$, then $g$ is weakly confluent with respect to $K$.

**Proof.** Let $g$ be a mapping of a continuum $M$ onto $X$, $n_1, n_2, n_3, \cdots$ be a sequence of integers, and $H_1, H_2, \cdots$ be a sequence of subcontinua of $M$ such that $\pi_i \circ g(H_i) = \pi_i(K_i)$ for $i = 1, 2, \cdots$. We can assume that the sequence $H_1, H_2, H_3, \cdots$ converges to a continuum $H$ in $M$.

We show that $K \subset g(H)$. Suppose $p \in K$ and $\varepsilon > 0$. Let $N$ be a positive integer such that if $k > N$ then $\sum_{i > N} 2^{-i} < \varepsilon$. Let $k > N$. Since $x \in K$, $\pi_n(K) = \pi_n(H_k)$. Let $x$ be a point of $g(H_k)$ such that $\pi_n(x) = \pi_n(p)$. Then for $i < n_k$, $\pi_i(x) = \pi_i(p)$. Thus, $d(p, x) = \sum_{i > 0} 2^{-i} d_1(\pi_i(x), \pi_i(p)) < \varepsilon$, and so $d(p, g(H_k)) < \varepsilon$ for $k > N$. Hence, $p \in \lim_{k \to \infty} g(H_k) = g(H)$. This shows that $K \subset g(H)$.

We show $g(H) \subset K$. Suppose $t \in g(K)$ and $\varepsilon > 0$. Let $N$ be a positive integer such that if $k > N$ then $\sum_{i > n_k} 2^{-i} < \frac{\varepsilon}{2}$. Choose $k > N$ such that $g(H) \subset B(g(H_k), \frac{\varepsilon}{2})$ and let $y$ be a point of $f(H_k)$ such that $d(t, y) < \frac{\varepsilon}{2}$. Since $y \in f(H_k)$ then...
\[ \pi_{n_k}(y) \in \pi_{n_k} \circ g(H) = \pi_{n_k}(H) \]. There is a point \( s \) in \( H \) such that \( \pi_{n_k}(y) = \pi_{n_k}(s) \), and so for \( i < n_k \), \( \pi_i(y) = \pi_i(s) \).

Thus, \( d(y, s) = \sum_{i=0}^{\infty} 2^{-i} d_i(\pi_i(y), \pi_i(s)) < \frac{\varepsilon}{2} \), and \( d(t, s) \leq d(t, y) + d(y, s) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \). Since for each \( \varepsilon > 0 \) there is a point \( x \) in \( H \) such that \( d(x, s) < \varepsilon \), then \( s \in \overline{H} = H \). This shows that \( g(H) \subseteq K \).

We have shown that \( g(H) = K \), thus \( g \) is weakly confluent with respect to \( K \).

In the following lemma, \( d \) denotes the Hausdorff metric.

**Lemma 2.** Suppose \( X \) is a continuum, \( K \) is a subcontinuum of \( X \), and \( g \) is a mapping of a continuum onto \( X \). If for each positive number \( \varepsilon \) there is a subcontinuum \( L \) of \( X \) such that \( g \) is weakly confluent with respect to \( L \) and \( d(K, L) < \varepsilon \), then \( g \) is weakly confluent with respect to \( K \).

**Proof.** The proof of this lemma is straightforward.

The next two theorems follow easily from the lemmas and Theorems 4 and 6 of section 2.

**Theorem 9.** Suppose \( X \) is the inverse limit of the inverse limit sequence \( \{X_i, f_i\} \) with each \( X_i \) a one-dimensional connected polyhedron, and \( K_1, \ldots, K_n \) are non-degenerate subcontinua of \( X \) such that for infinitely many integers \( i \),

1. the interiors of \( \pi_i K_1, \ldots, \pi_i K_n \) are mutually exclusive,
2. no one of \( \pi_i K_1, \ldots, \pi_i K_n \) contains a junction point of \( X_i \) in its interior, and
3. \( X_i - \bigcup_{j=1}^{n} \text{Int}(\pi_i K_j) \) is not connected. If \( g \) is a mapping of a continuum onto \( X \) then \( g \) is weakly confluent with respect to one of \( K_1, \ldots, K_n \).
Theorem 10. Suppose $X$ is the inverse limit of the inverse limit sequence $\{X_i, f_i\}$ with each $X_i$ a one-dimensional connected polyhedron, and $n$ is a positive integer such that $b_1(X_i) \leq n$ for each $i$. Suppose also that $K_1, \ldots, K_{n+1}$ are non-degenerate subcontinua of $X$ such that for infinitely many integers $i$, (1) the interiors of $\pi_i K_1, \ldots, \pi_i K_{n+1}$ are mutually exclusive and (2) no one of $\pi_i K_1, \ldots, \pi_i K_{n+1}$ contains a junction point of $X_i$ in its interior. If $g$ is a mapping of a continuum onto $X$ then $g$ is weakly confluent with respect to one of $K_1, \ldots, K_{n+1}$.

A special case of Theorem 9 was proved by Read [7, Theorem 4]. Theorems 9 and 10 may be used to show that certain one-dimensional continua are in Class(W). The following are continua for which Theorem 9 or 10 can be used to show they are in Class(W):

(1) the Class(W) continua defined by Waraszkiewicz in [9] (not all of the continua he described are in Class(W)),
(2) the Case-Chamberlin continuum [1],
(3) Ingram's continua in [3], and
(4) the continuum defined by Sherling in [8].

As an example, we will use Theorem 9 to show that the Case-Chamberlin continuum is in Class(W).

The Case-Chamberlin continuum (see [1]) is an inverse limit on figure eights using one bonding map. Let $A$ and $B$ be two circles tangent at a point $J$. Assign an orientation to each of $A$ and $B$. Let $f$ be a mapping which throws $A \cup B$ onto $A \cup B$ as follows:
(1) A is thrown onto $A \cup B$ by fixing $J$, then wrapping around $A$ in the positive direction, then $B$ in the positive direction, and then around each of $A$ and $B$ in the negative direction.

(2) $B$ is thrown onto $A \cup B$ by fixing $J$, then wrapping around $A$ twice in the positive direction, then $B$ twice in the positive direction, and then around each of $A$ and $B$ twice in the negative direction.

For each $i$ let $X_i = A \cup B$ and $f_i = f$. Let $X$ be the inverse limit of the inverse limit sequence $\{X_i, f_i\}$. One can show that if $K$ is a proper subcontinuum of $X$ then there exists a positive integer $n$ such that (1) for each $i > n$, $J \not\in \pi_n K$, or (2) for each $i > n$, $\pi_n K$ is an arc in $A$ having $J$ as an endpoint.

We will show that $X$ is in Class(W). Let $g$ be a mapping of a continuum onto $X$ and let $K$ be a proper subcontinuum of $X$. We will show that for every positive number $\varepsilon$ there is a subcontinuum $L$ of $X$ such that $g$ is weakly confluent with respect to $L$ and $d(K,L) < \varepsilon$ (where $d$ denotes the Hausdorff metric).

We assume $\prod (X_i, d_i)$ metrized by $d(x,y) = \sum_{i>0} 2^{-i}d(\pi_i x, \pi_i y)$. Let $\varepsilon > 0$ and $N$ be a positive integer such that $\sum_{i>N} 2^{-i} \frac{\varepsilon}{\text{diam}(A \cup B)}$. There exists an integer $N > J$ such that $J$ is not in the interior of $\pi_n K$.

We can choose mutually exclusive arcs $\alpha$ and $\beta$ in $A$ such that $f(\alpha) = f(\beta) = \pi_n K$ and $J$ is not in the interior of $\alpha$ or $\beta$. There exist subcontinua $L_1$ and $L_2$ of $X$ such that
\( \pi_{n+1}(L_1) = \alpha, \pi_{n+1}(L_2) = \beta \) and for each \( i > n+1 \), \( \pi_i(L_1) \) and \( \pi_i(L_2) \) are mutually exclusive arcs in \( A \), neither of which contains \( J \) in its interior. Then for \( i > n \),

\[
X_i = [\pi_i(L_1) \cup \pi_i(L_2)] \text{ is not connected.}
\]

By Theorem 9, \( g \) is weakly confluent with respect to \( L_1 \) or \( L_2 \). Since \( \pi_n(L_1) = \pi_n(L_2) = \pi_n(K) \), then

\[
d(K,L_1) < \sum_{i<n} 2^{-i} (\text{diam } A \cup B) < \epsilon, \quad \text{and}
\]

\[
d(K,L_2) < \sum_{i<n} 2^{-i} (\text{diam } A \cup B) < \epsilon.
\]

Therefore, for each positive number \( \epsilon \) there is a subcontinuum \( L \) of \( X \) such that \( g \) is weakly confluent with respect to \( L \) and \( d(K,L) < \epsilon \). By Lemma 2, \( g \) is weakly confluent with respect to \( K \). Hence, \( X \) is in Class(W).

References


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