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1. Introduction

We present an easy proof of the result that every locally compact Hausdorff space with a weakly uniform base is metrizable [P].

A collection $\mathcal{G}$ of subsets of a space X is a weakly uniform family provided that if $\mathcal{H}$ is any infinite subcollection of $\mathcal{G}$, then $|\cap\mathcal{H}| \leqq 1$. A weakly uniform base for a space X is a base for X which is a weakly uniform family [HL].

We shall need the following lemmas. We assume that all spaces are Hausdorff.

Lemma 1 [HL]. *If* X *has a weakly uniform base, then* X *has a* G_δ*-diagonal.*

Lemma 2 [S]. *If* X *is compact and has a* G_δ*-diagonal, then* X *is metrizable.*

Lemma 3 [HL]. *If* $\mathcal{B}$ *is a weakly uniform base for a separable space* X*, then* $\mathcal{B}$ *is countable.*

Lemma 4 [HL]. *If* x *is a non-isolated point of a first countable space* X *which has a weakly uniform base* $\mathcal{B}$*, then* $\mathcal{B}$ *has countable order at* x.

Lemma 5 [CM]. *If* X *is a locally compact space with a point-countable base, then* X *is metrizable.*

Lemma 6 [DRW]. *If* X *is a regular first countable space and* $\mathcal{G}$ *is a weakly uniform family of open sets covering* E, *the set of all nonisolated points in* X, *such that*

(1) *each element of* $\mathcal{G}$ *contains at most one point of* E, *and*

(2) *for each* $D \subseteq E$, $|M| \leq |D| + \omega$ *where* M *is the set of all nodes of* $\mathcal{G}(D) = \{G \in \mathcal{G}: G \cap D \neq \emptyset\}$,

then $\mathcal{G}$ *has a point-finite open partial refinement covering* E.

In Lemma 6, a *node* of a collection of sets is a point which is in more than one element of the collection.

2. The Result

We now proceed to the proof of the theorem. This result was obtained by Peregudov in [P] by a much more elaborate argument.

Theorem. Every locally compact space with a weakly uniform base is metrizable.

Proof. Suppose X is a locally compact space with a weakly uniform base $\mathcal{B}$. For each $x \in X$, choose open U_x with $x \in U_x$ and $\overline{U}_x$ compact. The collection $\{B \in \mathcal{B}: B \subseteq U_x$ for some $x\}$ is a weakly uniform base for X, so WLOG we assume that $\mathcal{B}$ is made up of sets with compact closures. Since X has a G_δ-diagonal, we have that for each $B \in \mathcal{B}$, $\overline{B}$ is a compact metric space. Hence B is separable, and by Lemma 3, $\{G \in \mathcal{B}: G \subseteq B\}$ is a countable base for B.

Let E denote the nonisolated points of X. Then $\mathcal{B}' = \{E \cap B: B \in \mathcal{B}\}$ is a point-countable base for the locally compact space E. Thus E is metric by Lemma 5. Choose a σ-discrete open cover $\cup_{n\in\omega}\mathcal{U}_n$ of E such that for each n, $\mathcal{U}_n$ is a discrete collection of nonempty open sets in E and is a precise partial refinement of $\mathcal{B}'$. For $U \in \mathcal{U}_n$, choose the $B \in \mathcal{B}$ corresponding to U such that $U \subseteq E \cap B$. Let $V_U = U \cup (B - E)$. Then let $\mathcal{V}_n = \{V_U: U \in \mathcal{U}_n\}$. For each $U \in \mathcal{U}_n$, $\{B \in \mathcal{B}: \overline{B} \subseteq V_U\}$ is a countable open cover of V_U, and we index it by $\{B(U,k): k \in \omega\}$. For each $n \in \omega$, $k \in \omega$, let $X_{n,k} = \{\overline{B(U,k)} \cap E: U \in \mathcal{U}_n\} \cup (X - E)$ with the quotient topology.

For each n,k, since $\overline{B(U,k)} \cap E$ is compact and V_U is a second countable space, we have that $X_{n,k}$ is a regular first countable space, and $\{\{\overline{B(U,k)} \cap E\} \cup (V_U - E): U \in \mathcal{U}_n\}$ is a weakly uniform open cover of the non-isolated points of $X_{n,k}$. Let $\mathcal{W}'_{n,k}$ be a point-finite open partial refinement which covers the non-isolated points by Lemma 6, say $\mathcal{W}'_{n,k} = \{\{\overline{B(U,k)} \cap E\} \cup W(U,k): U \in \mathcal{U}_n\}$. Then $\mathcal{W}_{n,k} = \{(B(U,k) \cap E) \cup W(U,k): U \in \mathcal{U}_n\}$ is an open (in X) partial refinement of $\mathcal{V}_n$ which is point-finite in X and covers $\cup\{B(U,k) \cap E: U \in \mathcal{U}_n\}$. Now $\cup_{n,k}\mathcal{W}_{n,k}$ is a point countable collection of second countable open subsets of X, and

$$\{B \in \mathcal{B}: B \subseteq W \text{ for some } W \in \cup_{n,k}\mathcal{W}_{n,k}\} \cup \{\{x\}: x \notin \cup\cup_{n,k}\mathcal{W}_{n,k}\}$$

is a point-countable base for X. Hence, by Lemma 5, X is metrizable.

Remark. Note that it follows immediately from the above theorem and Corollary 6 of [HL] that every locally countably compact regular space with a weakly uniform base is metrizable.

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