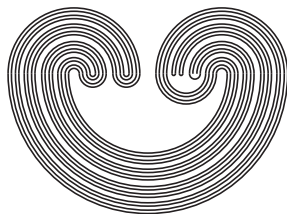

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ON A THEOREM OF PEREGUDOV

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ON A THEOREM OF PEREGUDOV**D. K. Burke and S. W. Davis****1. Introduction**

We present an easy proof of the result that every locally compact Hausdorff space with a weakly uniform base is metrizable [P].

A collection \mathcal{G} of subsets of a space X is a weakly uniform family provided that if \mathcal{H} is any infinite subcollection of \mathcal{G} , then $|\cap \mathcal{H}| \leq 1$. A weakly uniform base for a space X is a base for X which is a weakly uniform family [HL].

We shall need the following lemmas. We assume that all spaces are Hausdorff.

Lemma 1 [HL]. If X has a weakly uniform base, then X has a G_δ -diagonal.

Lemma 2 [S]. If X is compact and has a G_δ -diagonal, then X is metrizable.

Lemma 3 [HL]. If \mathcal{B} is a weakly uniform base for a separable space X , then \mathcal{B} is countable.

Lemma 4 [HL]. If x is a non-isolated point of a first countable space X which has a weakly uniform base \mathcal{B} , then \mathcal{B} has countable order at x .

Lemma 5 [CM]. If X is a locally compact space with a point-countable base, then X is metrizable.

Lemma 6 [DRW]. If X is a regular first countable space and \mathcal{G} is a weakly uniform family of open sets covering E , the set of all nonisolated points in X , such that

(1) each element of \mathcal{G} contains at most one point of E , and

(2) for each $D \subseteq E$, $|M| \leq |D| + \omega$ where M is the set of all nodes of $\mathcal{G}(D) = \{G \in \mathcal{G} : G \cap D \neq \emptyset\}$,

then \mathcal{G} has a point-finite open partial refinement covering E .

In Lemma 6, a *node* of a collection of sets is a point which is in more than one element of the collection.

2. The Result

We now proceed to the proof of the theorem. This result was obtained by Peregudov in [P] by a much more elaborate argument.

Theorem. Every locally compact space with a weakly uniform base is metrizable.

Proof. Suppose X is a locally compact space with a weakly uniform base β . For each $x \in X$, choose open U_x with $x \in U_x$ and \bar{U}_x compact. The collection $\{B \in \beta : B \subseteq U_x \text{ for some } x\}$ is a weakly uniform base for X , so WLOG we assume that β is made up of sets with compact closures. Since X has a G_δ -diagonal, we have that for each $B \in \beta$, \bar{B} is a compact metric space. Hence B is separable, and by Lemma 3, $\{G \in \beta : G \subseteq B\}$ is a countable base for B .

Let E denote the nonisolated points of X . Then $\beta' = \{E \cap B : B \in \beta\}$ is a point-countable base for the

locally compact space E . Thus E is metric by Lemma 5.

Choose a σ -discrete open cover $\cup_{n \in \omega} \mathcal{U}_n$ of E such that for each n , \mathcal{U}_n is a discrete collection of nonempty open sets in E and is a precise partial refinement of β' . For $U \in \mathcal{U}_n$, choose the $B \in \beta$ corresponding to U such that $U \subseteq E \cap B$. Let $V_U = U \cup (B - E)$. Then let $\mathcal{V}_n = \{V_U : U \in \mathcal{U}_n\}$. For each $U \in \mathcal{U}_n$, $\{B \in \beta : \overline{B} \subseteq V_U\}$ is a countable open cover of V_U , and we index it by $\{B(U,k) : k \in \omega\}$. For each $n \in \omega$, $k \in \omega$, let $X_{n,k} = \{\overline{B(U,k)} \cap E : U \in \mathcal{U}_n\} \cup (X - E)$ with the quotient topology.

For each n,k , since $\overline{B(U,k)} \cap E$ is compact and V_U is a second countable space, we have that $X_{n,k}$ is a regular first countable space, and $\{\{\overline{B(U,k)} \cap E\} \cup (V_U - E) : U \in \mathcal{U}_n\}$ is a weakly uniform open cover of the non-isolated points of $X_{n,k}$. Let $\mathcal{W}'_{n,k}$ be a point-finite open partial refinement which covers the non-isolated points by Lemma 6, say $\mathcal{W}'_{n,k} = \{\{\overline{B(U,k)} \cap E\} \cup W(U,k) : U \in \mathcal{U}_n\}$. Then $\mathcal{W}_{n,k} = \{(B(U,k) \cap E) \cup W(U,k) : U \in \mathcal{U}_n\}$ is an open (in X) partial refinement of \mathcal{V}_n which is point-finite in X and covers $\cup\{B(U,k) \cap E : U \in \mathcal{U}_n\}$. Now $\cup_{n,k} \mathcal{W}_{n,k}$ is a point countable collection of second countable open subsets of X , and

$$\{B \in \beta : B \subseteq W \text{ for some } W \in \cup_{n,k} \mathcal{W}_{n,k}\} \cup \{\{x\} : x \notin \cup_{n,k} \mathcal{W}_{n,k}\}$$

is a point-countable base for X . Hence, by Lemma 5, X is metrizable.

Remark. Note that it follows immediately from the above theorem and Corollary 6 of [HL] that every locally countably compact regular space with a weakly uniform base is metrizable.

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