AN ADMISSIBLE CONDITION FOR
CONTRACTIBLE HYPERSPACES

by

CHOOK JAI RHEE AND TOGO NISHIURA
Let $X$ be a nonvoid metric continuum. Denote by $2^X$ and $C(X)$ the hyperspaces of nonempty closed subsets and subcontinua of $X$ respectively and endow each with the Hausdorff metric $H$.

In 1938 Wojkyslawski proved that $2^X$ is contractible if $X$ is locally connected [11]. In 1942 Kelley [1] proved that the contractibility of $2^X$ is equivalent to the contractibility of $C(X)$. Furthermore, he introduced a sufficient condition, namely property (3.2), for the contractibility of the hyperspaces of metric continua. In [5], a necessary condition, namely admissibility, is given for a space whose hyperspace is contractible. It was also proven that the contractibility of the hyperspace $C(X)$ is equivalent to the existence of a continuous fiber map on $X$ into the hyperspace $C^2(X)$ of subcontinua of $C(X)$ for the class of metric continua with property c (abbreviated as c-space).

In the present paper we show that if $f: X \to Y$ and $g: Y \to X$ are continuous such that $f \circ g$ is homotopic to the identity map $id_Y$ on $Y$, and if $X$ is a c-space then $Y$ is also a c-space. Hence if $X$ and $Y$ are homotopically equivalent, then $X$ is a c-space if and only if $Y$ is. We also show that the product space $X \times Y$ is a c-space if and only if both $X$ and $Y$ are c-spaces. Many corollaries to the above results are also given which are generalizations of results in [4].
Throughout the paper, the symbols $I$ and $\mu$ will be reserved for the closed interval and a Whitney map [10] with $\mu(X) = 1$ respectively. Note that $\mu(X) = 1$ necessarily requires $X$ to be nondegenerate, a condition which we will assume whenever required without explicitly stating so.

1. Preliminaries

We collect in this section some definitions and known facts and prove a new lemma. Let $X$ be a nonvoid metric continuum.

A map $H: X \times I \to C(X)$ is increasing if $h(x,t) \subseteq h(x,t')$ for $t \leq t'$ and $x \in X$. A contraction of $X$ in $C(X)$ is a continuous homotopy $h: X \times I \to C(X)$ such that, for each $x \in X$, $h(x,0) = \{x\}$ and $h(x,1) = A$. A contraction of $X$ in $2^X$ is analogously defined.

**Theorem 1.1** [1]. The following statements are equivalent.

1. A contraction of $X$ in $C(X)$ exists.
2. $2^X$ is contractible.
3. $C(X)$ is contractible.

**Theorem 1.2** [1]. If $C(X)$ is contractible then an increasing contraction of $X$ in $C(X)$ exists.

The contractibility of $C(X)$ implies the contractibility of $C^2(X)$, by Theorem 1.1. Also, since the union may from $C^2(X)$ onto $C(X)$ is a retraction, the contractibility of $C^2(X)$ implies the contractibility of $C(X)$. Thus we have the following.
Theorem 1.3 \[1\]. \(C(X)\) is contractible if and only if the hyperspace \(C^2(X)\) of subcontinua of \(C(X)\) is contractible.

We recall the definition of the Hausdorff metric \(H\) on \(2^X\). For \(A,B \in 2^X\),
\[
H(A,B) = \max\{\max_{a \in A} d(a,B), \max_{b \in B} d(b,A)\},
\]
where \(d(x,A)\) is the distance from \(x\) to \(A\).

Lemma 1.4. If \(A,B,C,D \in 2^X\) then
\[
H(A \cup B, C \cup D) \leq \max\{H(A,C), H(B,D)\}.
\]

Proof. Let \(\eta > \max\{H(A,C), H(B,D)\}\). Then \(C \cup D \subseteq \{x | d(x,A) < \eta\} \cup \{x | d(x,B) < \eta\} = \{x | d(x,A \cup B < \eta\}\).

Let \(X\) be a nonvoid continuum. We now define an admissibility condition [5] and prove some propositions.

For \(x \in X\), let \(F(x) = \{A \in C(X) | x \in A\}\), and for \((x,t) \in X \times I\),
\[
F_t(x) = F(x) \cap \mu^{-1}(t).
\]
An element \(A \in F(x)\) is said to be admissible at \(x\) if, for each \(\varepsilon > 0\), there is \(\delta > 0\) such that each \(y\) in the \(\delta\)-neighborhood of \(x\) has an element \(B \in F(y)\) such that \(H(A,B) < \varepsilon\). For each \(x \in X\), the collection \(A(x) = \{A \in F(x) | A\) is admissible at \(x)\) is called the admissible fiber at \(x\). We say that \(X\) is admissible if \(A_t(x) = A(x) \cap \mu^{-1}(t)\) is nonempty for each \((x,t) \in X \times I\).

Proposition 1.5. If \(A \in A(\xi)\) and \(B \in A(x)\) and \(\xi \in A \cap B\) then \(A \cup B \in A(x)\). Hence, if \(A_i \in A(x)\), \(i = 1,2,\ldots,n\), then \(\bigcup_{i=1}^n A_i \in A(x)\).

Proof. Let \(\varepsilon > 0\). Since \(A \in A(\xi)\), there is \(\tau < \varepsilon\) where each point \(y\) of the \(\tau\)-neighborhood \(V\) of \(\xi\) has an element \(C \in F(y)\) such that \(H(A,C) < \varepsilon\). Since \(B \in A(x)\) there is
\( \delta > 0 \) such that each point \( z \) of the \( \delta \)-neighborhood \( W \) of \( x \) has an element \( D \in F(z) \) such that \( H(B,D) < \tau \). One sees that \( \xi \in B \) and \( H(B,D) < \tau \) imply \( V \cap D \neq \emptyset \). Hence, for each \( z \in W \) there are \( D \in F(z) \), \( y \in V \cap D \) and \( C \in F(y) \) such that \( H(A,C) < \varepsilon \), \( H(B,D) < \varepsilon \) and \( C \cup D \in F(z) \). By Lemma 1.4, we have \( H(A \cup B,C \cup D) \leq \max\{H(A,C),H(B,D)\} < \varepsilon \), and the proposition is proved.

**Proposition 1.6.** For each \( x \in X \) its admissible fiber \( A(x) \) is closed in \( C(X) \), \( \{x\} \in A(x) \) and \( x \in A(x) \).

**Proof.** Suppose \( A_n \), \( n = 1,2,\ldots \), is a sequence in \( A(x) \) which converges to \( A \) in \( C(X) \). Obviously, \( A \in F(x) \). Let \( \varepsilon > 0 \). There is a positive integer \( N \) such that \( H(A,A_N) < \varepsilon/2 \). Since \( A_N \in A(x) \), there is a \( \delta \)-neighborhood \( V \) of \( x \) such that each point \( y \) of \( V \) has an element \( B \in F(y) \) such that \( H(A,N,B) < \varepsilon/2 \). From \( H(A,B) \leq H(A,A_N) + H(A_N,B) < \varepsilon \), we have \( A \in A(x) \) and hence \( A(x) \) is closed. The remaining parts of the proposition are obvious.

We note that since \( C(X) \) is compact [1], \( A(x) \) is compact.

**Proposition 1.7.** Let \( B \in F(x) \) and \( C = \cup\{A \in A(x) \mid A \subset B\} \) then \( C \in A(x) \).

**Proof.** First we prove \( C \) is a subcontinuum of \( X \). Clearly \( C \) is connected and \( x \in C \). Let \( x_n \), \( n = 1,2,\ldots \), be a sequence in \( C \) converging to \( x_0 \). For each \( n \geq 1 \) choose \( A_n \in A(x) \) such that \( x_n \in A_n \subset B \). Since \( A(x) \) is compact in \( C(X) \), we may assume that the sequence \( A_n \), \( n = 1,2,\ldots \), also converges to an element \( A_0 \in A(x) \). Obviously, \( x_0 \in A_0 \subset B \). Hence \( x_0 \in A_0 \subset C \). We conclude that \( C \) is closed in \( X \).
Now suppose $\varepsilon > 0$. Since $C$ is compact in $X$, there are points $c_1, c_2, \cdots, c_n$ in $C$ such that $C$ is contained in the $\varepsilon$-neighborhood of the finite set $\{c_1, c_2, \cdots, c_n\}$. For each $i$, let $A_i \in A(x)$ such that $c_i \in A_i \subset B$ and let $B_0 = \bigcup_{i=1}^{n} A_i$. Since $C \supset B_0 \supset \{c_1, c_2, \cdots, c_n\}$, we have $H(C, B_0) < \varepsilon$. By Proposition 1.5, $B_0 \in A(x)$. Since $A(x)$ is compact in $C(X)$ we have $C \in A(x)$.

Proposition 1.8 [7]. If $h: X \times I \to C(X)$ is a continuous increasing map such that $x \in h(x, 0)$ for $x \in X$ then $h(x, t) \in A(x)$ for $(x, t) \in X \times I$.

Theorem 1.9 [7]. If $X$ is a nondegenerate metric continuum and $C(X)$ is contractible, then $X$ is an admissible space.

2. Fiber Maps

In [5] it was shown that the contractibility of $C(X)$ is equivalent to an existence of a set-valued map $\alpha: X \to C(X)$ possessing a certain property. In this section we prove that this property is preserved by the homotopy equivalence relation. Hence, we obtain generalizations of many of the results in [4] and [8].

Definition 2.1 [5]. A set-valued map $\alpha: X \to C(X)$ is said to be a c-map if, for each $x \in X$, $\alpha(x)$ is a closed subset of the admissible fiber $A(x)$ such that

(1) $\{x\}, X \in \alpha(x)$.

(2) For each pair $A_0, A_1$ in $\alpha(x)$ with $A_0 \subset A_1$, there is an ordered segment [2, p. 57] in $\alpha(x)$ from $A_0$ to $A_1$. 

(3) For each $A \in \alpha(x)$, and $\varepsilon > 0$, there is a neighborhood $W$ of $x$ such that each point $y$ of $W$ has an element $B \in \alpha(y)$ such that $H(A, B) < \varepsilon$.

We say that the space $X$ is a $c$-space if there is a set-valued $c$-map $\alpha: X \to C(X)$. Clearly every $c$-space is an admissible space.

**Proposition 2.2** [5]. Every set-valued $c$-map $\alpha: X \to C(X)$ is lower semicontinuous. Furthermore, if $\hat{\alpha}(x, t) = \alpha(x) \cap \mu^{-1}(t)$, then $\hat{\alpha}$ is lower semicontinuous on $X \times I$.

**Theorem 2.3** [5]. Let $X$ be a metric continuum. Then $C(X)$ is contractible if and only if there is continuous set-valued $c$-map on $X$ into $C(X)$.

In [1] Kelley defined a property (subsequently named property K in Nadler [2]) and proved that the hyperspaces of a space having property K are always contractible. The class of metric continua having property K includes locally connected continua and the hereditarily indecomposable continua. We now restate the result of Kelley.

**Proposition 2.4** [1]. If $X$ has property K, then there is a continuous $c$-map $\alpha: X \to C(X)$.

**Proof.** Since $X$ has property K, $F(x) = \hat{A}(x)$ by [5, Proposition 2.4] and $F: X \to C(X)$ is continuous by [9, Theorem 2.2]. The existence of ordered segments in $F(x)$ for every pair $A_0 \subseteq A_1$ is given in [1, p. 24]. Hence the admissible fiber map $\hat{A}$ is a continuous set-valued $c$-map.
Let $X$ be a metric continuum. A function $\alpha: X \to C^2(X)$ is called admissible if, for each $x \in X$,

1. $\{x\} \in \alpha(x)$,
2. $\alpha(x) \subseteq \mathcal{A}(x)$ and $\alpha(x)$ is closed in $\mathcal{A}(x)$,
3. $\alpha(x)$ contains a maximal element $A_x$, i.e., $A \subseteq A_x$ for all $A \in \alpha(x)$,
4. $\alpha(x)$ is segmentwise connected, i.e., for each pair $A_0, A_1$ in $\alpha(x)$ with $A_0 \subseteq A_1$, there is an ordered segment [2, p. 57] in $\alpha(x)$ from $A_0$ to $A_1$.

Let $n_\alpha = \{A_x | A_x$ is a maximal element in $\alpha(x), x \in X\}$ and $N_\alpha^2 = \{\{A_x^2 | A_x^2 \in N_\alpha\} \subseteq C^2(X)$.

**Proposition 2.5.** The following statements are equivalent.

1. $C(X)$ is contractible.
2. There is a continuous admissible function $\alpha: X \to C^2(X)$ such that $\cap N_\alpha \neq \emptyset$.
3. There is a continuous admissible function $\alpha: X \to C^2(X)$ such that the set $N_\alpha^2$ is contractible in $C^2(X)$.

**Proof.** (1) $\Rightarrow$ (2). Suppose $h: X \times I \to C(X)$ is an increasing contraction. Let $\alpha(x) = \{h(x,t) | t \in I\}$. Then the continuity of $h$ provides the continuity of $\alpha$ and it is obvious that $\alpha$ satisfies the admissible conditions (1')-(4'), with $A_x = h(x,1) = \cap N_\alpha$.

(2) $\Rightarrow$ (3). Suppose $\alpha: X \to C^2(X)$ is a continuous admissible function such that $\cap N_\alpha \neq \emptyset$. Let $x_0 \in \cap N_\alpha$ and let $\gamma: I \to C(X)$ be an ordered segment from $\{x_0\}$ to $X$. Define $\beta: N_\alpha^2 \times I \to C^2(X)$ by $\beta(A_x \gamma(t), t) = A_x \cup \gamma(t)$.
Then $\beta$ is continuous and $\beta({A_x},1) = \{x\}$ for each
$\{A_x\} \in \mathcal{N}_u^2$.

(3) $\Rightarrow$ (1). Suppose $\alpha: X \to C^2(X)$ is a continuous admissible function and $\beta: \mathcal{N}_u^2 \times I \to C^2(X)$ is a contraction. We may assume $\beta$ is increasing. Let $\sigma: C^2(X) \to C(X)$ be the function defined by $\sigma(T) = UT$. Then $\sigma$ is continuous.

We now observe that for each $x \in X$, the maximal element $A_x$ of $\alpha(x)$ is unique and $A_x = \sigma \circ \alpha(x)$. Therefore the function $x \to \{A_x\}$ is continuous from $X$ to $C^2(X)$. Hence we define a function $\tau: X \to C^2(X)$ by $\tau(x) = \alpha(x) \cup \{\sigma(\beta({A_x},t))|t \in I\}$. Then $\tau$ is continuous. Since $\sigma(\beta({A_x},1)) = A_x$, for some $A$, and for all $x \in X$, we may define an ordered segment $\gamma: I \to C(X)$ from $A$ to $X$ and join it to $\tau$, that is $\phi(x) = \tau(x) \cup \{\gamma(t)|t \in I\}$. Then it is not difficult to check that $\phi$ satisfies the definition of a set-valued map and $\phi$ is continuous. Hence by Theorem 2.3, $C(X)$ is contractible.

Suppose $X$ and $Y$ are metric continua.

**Theorem 2.6.** Suppose $f: X \to Y$ and $g: Y \to X$ are continuous functions such that $f \circ g: Y \to Y$ is homotopic to the identity $id_Y$. If $X$ is a c-space then $Y$ is a c-space.

**Proof.** Let $h: Y \times I \to Y$ be a homotopy such that $h(y,0) = y$ and $h(y,1) = f \circ g(y)$ for each $y \in Y$. Let $\overline{h}(y,t) = \cup \{h(y,s)|0 \leq s \leq t\}$. Then $\overline{h}: Y \times I \to C(Y)$ is a continuous homotopy such that $\overline{h}(y,t) \subset \overline{h}(y,t')$ whenever $t \leq t'$. Let $\beta_1(y) = \{\overline{h}(y,t)|t \in I\}$. Then the continuity of $\overline{h}$ implies the continuity of $\beta_1: Y \to C^2(Y)$ and each element $\overline{h}(y,t)$ of the set $\beta_1(y)$ is an admissible element at $y$ such
that for each pair $B_0, B_1$ in $\beta_1(y)$ with $B_0 \subset B_1$, there is an ordered segment in $\beta_1(y)$ from $B_0$ to $B_1$.

Let $a: X \to C(X)$ be a set-valued c-map. For $y \in Y$, let $x = g(y)$ and $\beta_2(y) = \{h(y,l) \cup f(A) | A \in a(x)\}$. Since $f(x) = f \circ g(y) \in h(y,l) \cap f(A)$, we have $h(y,l) \cup f(A) \in C(Y)$. Now we will show that $\beta_2: Y \to C^2(Y)$ is lower semicontinuous.

Let $\epsilon > 0$. Since $f$ is continuous, there is $\epsilon' > 0$ such that if $A, A' \in C(X)$ such that $A$ and $A'$ are less than $\epsilon'$ apart, then $H(f(A), f(A')) < \epsilon$. Since $a$ is lower semicontinuous and $a(x)$ is compact, there exists $\delta > 0$ such that if $d(x, x') < \delta$ and $A \in a(x)$, there is an element $A' \in a(x')$ such that $A$ and $A'$ are less than $\epsilon'$ apart. Now the continuity of $g$ implies that there is $\delta_0 > 0$ such that if $d(y, y') < \delta_0$ then $d(g(y), g(y')) < \delta$. Also, by the continuity of $h$, we choose $\delta_1 > 0$ such that if $d(y, y') < \delta_1$, then $H(h(y,l), h(y', l)) < \epsilon'$. Let $\delta = \min\{\delta_0, \delta_1\}$, $x = g(y)$, $x' = g(y')$. Then if $d(y, y') < \delta$ and $h(y,l) \cup f(A) \in \beta_2(y)$, then there is $h(y',l) \cup f(A') \in \beta_2(y')$ such that $H(h(y,l) \cup f(A), h(y',l) \cup f(A')) < \epsilon$ by Lemma 1.4. Hence the elements of $\beta_2(y)$ are admissible at $y$ and $\beta_2$ is lower semicontinuous. Since $f$ preserves ordered segments, we see that $\beta_2$ satisfies the condition of ordered segment. Now let $\gamma: I \to C(Y)$ be an ordered segment from $f(X)$ to $Y$ and let $\beta_3(y) = \{\gamma(t) \cup h(y,l) | t \in I\}$ and $\beta(y) = \beta_1(y) \cup \beta_2(y) \cup \beta_3(y)$. The maximal element of $\beta_1(y)$ is $h_1(y,l)$ which is also the minimal element of $\beta_2(y)$, and the maximal element of $\beta_2(y)$ is $h(y,l) \cup f(X)$, and the minimal element of $\beta_3(y)$ is $f(X) \cup h(y,l)$. So the continuity of $\beta_1$
and $\beta_3$ together with the lower semicontinuity of $\beta_2$ provide the lower semicontinuity of $\beta$, and hence (3) is verified for $\beta$. It is easy to verify that $\beta$ also satisfies the condition (1) and (2) of Definition 2.1.

**Corollary 2.7.** Suppose $X$ and $Y$ are homotopically equivalent metric continua. Then $X$ is a c-space if and only if $Y$ is.

Let $\text{id}_Y$ denote the identity map of $Y$ onto itself and $[f]$ the homotopy class of continuous maps of $Y$ into itself which contains $f$.

**Theorem 2.8.** A metric continuum $Y$ is a c-space if and only if for some $g$ in $[\text{id}_Y]$ it is true that $g(Y)$ is a c-space.

**Proof.** If $Y$ is a c-space then let $g = \text{id}_Y$. Conversely, suppose for some $g \in [\text{id}_Y]$, $g(Y)$ is a c-space. Let $X = g(Y)$ and $f: X \to Y$ be the inclusion map. Then $f \circ g = g \in [\text{id}_Y]$. So Theorem 2.6 gives the conclusion.

**Corollary 2.9.** Suppose $X$ is a deformation retract of $Y$. If $X$ is a c-space, so is $Y$.

**Proof.** Let $\gamma: Y \to X$ be a retraction which is homotopic to the identity map $\text{id}_Y$. Then Theorem 2.8 provides the conclusion.

**Corollary 2.10.** If $Y$ is a retract of $X$ and $X$ is a c-space, then so is $Y$. 
Proof. Let \( f: X \rightarrow Y \) be a retraction and \( g: Y \rightarrow X \) the inclusion map. Then \( f \circ g = \text{id}_Y \). So by Theorem 2.6, \( Y \) is a c-space.

Theorem 2.11. Let \( X = X_1 \cup X_2 \) where \( X \) and \( X_1 \) are subcontinua and \( X_2 \) is a closed subset such that \( X_1 \cap X_2 \) is a strong deformation retract of \( X_2 \). If \( X_1 \) is a c-space so is \( X \).

Proof. \( X_1 \) is a deformation retract of \( X \).

Theorem 2.12. The product space \( X \times Y \) is a c-space if and only if both \( X \) and \( Y \) are c-spaces.

Proof. Each factor space is a retract of \( X \times Y \).

Therefore by Corollary 2.10, \( X \times Y \) is a c-space.

Conversely, if \( \alpha_X: X \rightarrow C(X) \) and \( \alpha_Y: Y \rightarrow C(Y) \) are set-valued c-maps, then \( \alpha_X \times \alpha_Y: X \times Y \rightarrow C(X \times Y) \) defined by

\[
(\alpha_X \times \alpha_Y)(x,y) = \alpha_X(x) \times \alpha_Y(y) = \{A \times B | A \in \alpha_X(x), B \in \alpha_Y(y)\}
\]

is a set-valued c-map.

References


Wayne State University

Detroit, Michigan 48202