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Robert Buskirk and Sam B. Nadler, Jr.

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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
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1. Introduction

Let (X,T) be a topological space. An *extension* of T is a topology T' for X such that $T' \supset T$. If α is a collection of subsets of X, then the *extension of* T *by* α, denoted by $T(\alpha)$, is the topology for X having $T \cup \alpha$ as a subbase. It has been of interest to find conditions on the members of α in order that a given topological property be preserved under the extension of T by α (see [1], [3]-[9]). In this paper we give a necessary and sufficient condition on the members of α in order that $(X,T(\alpha))$ be weakly equivalent to (X,T). As a consequence, we have a sufficient condition on the members of α so that dense sets, residual sets, separability, the property of being a Baire space, etc. are preserved under the extension of T by α. We also obtain conditions which imply that a $T(\alpha)$-continuous function into a metric space is T-continuous at each point of a T-dense set.

2. Weakly Equivalent Extensions

Two topologies T and T* for a set X are said to be *weakly equivalent* provided that each nonempty member of either of the topologies contains a nonempty member of the other. We say that a collection α of subsets of a set X has *property* (W) *with respect to a topology* T for X provided that for each finite subcollection α' of α,

$$cl_T[int_T(\cap\alpha')] \supset \cap\alpha'$$

where cl_T and int_T denote the closure and interior operators with respect to T.

Theorem 2.1. *Let* (X,T) *be a topological space and let* α *be a collection of subsets of* X. *Then, the topologies* T *and* T(α) *are weakly equivalent if and only if* α *has property* (W) *with respect to* T.

Proof. First, suppose that α does not have property (W) with respect to T. Then, there is a finite subcollection α' of α such that

$$cl_T[int_T(\ \alpha')] \not\supset \cap\alpha'.$$

Let $G = (X - cl_T[int_T(\cap\alpha')]) \cap (\cap\alpha')$. Note that $G \neq \phi$ and $G \in T(\alpha)$. Suppose that there is a $V \in T$ such that $V \neq \phi$ and $V \subset G$. Since $V \subset G$, $V \subset \cap\alpha'$. Thus, since $V \in T$, $V \subset int_T(\cap\alpha')$. However, since $V \subset G$, $V \subset X - int_T(\cap\alpha')$ which, since $V \neq \phi$, is a contradiction. Hence, G does not contain any nonempty member of T. Therefore, T and T(α) are not weakly equivalent. Conversely, assume that α has property (W) with respect to T. Let $H \in T(\alpha)$ such that $H \neq \phi$. Then, since $T \cup \alpha$ is a subbase for T(α), there is a finite subcollection α' of α and a $U \in T$ such that $U \cap (\cap\alpha')$ is a nonempty subset of H. Let $V = int_T(\cap\alpha')$. Since α has property (W) with respect to T, $cl_T(V) \supset \cap\alpha'$. Thus, since $U \in T$ and $U \cap (\cap\alpha') \neq \phi$, $U \cap V \neq \phi$. Clearly, $U \cap V \in T$ and $U \cap V \subset H$. Hence, we have proved that each nonempty member of T(α) contains a nonempty member of T. Therefore, since $T \subset T(\alpha)$, T and T(α) are weakly equivalent.

Let us note that there are collections α such that for any two members A and B of α

$$cl_T[int_T(A \cap B)] \supset A \cap B$$

and yet α fails to have property (W) with respect to T. For example: Let $X = \{(x,o) \in R^2: -1 \le x \le 1\} \cup \{(o,y) \in R^2: o \le y \le 1\}$ with the topology T inherited from the usual topology on the plane R^2, and let $\alpha = \{A_1, A_2, A_3\}$ where
$A_1 = \{(x,o) \in X: -1 \le x \le o\} \cup \{(o,y) \in X: o \le y \le 1\}$,
$A_2 = \{(x,o) \in X: o \le x \le 1\} \cup \{(o,y) \in X: o \le y \le 1\}$, and
$A_3 = \{(x,o) \in X: -1 \le x \le 1\}$.

3. Applications

First, recall the following definitions. Let (X,T) be a topological space and let $A \subset X$. Then: A is T-*dense* in X provided that $cl_T(A) = X$; A is T-*nowhere dense* in X provided that $int_T[cl_T(A)] = \phi$; A is T-*first category* in X provided that $A = \{A_i: i = 1,2,\cdots\}$ where each A_i is T-nowhere dense in X; A is T-*second category* in X provided that A is not T-first category in X; A is T-*residual* in X provided that X-A is T-first category in X. Fort [2] has observed that if two topologies on X are weakly equivalent, then a subset A of X has any of the properties above with respect to one of the topologies if and only if A has the same property with respect to the other topology. Thus, as a consequence of (2.1), we have the following result.

Theorem 3.1. *Let* (X,T) *be a topological space, let* α *be a collection of subsets of* X *having property* (W) *with respect to* T, *and let* $A \subset X$. *Then,* A *is* T-*dense,* T-*nowhere* *dense,* T-*first category,* T-*second category, or* T-*residual if and only if* A *is* T(α)-*dense,* T(α)-*nowhere dense,* T(α)-*first category,* T(α)-*second category, or* T(α)-*residual, respectively.*

Corollary 3.2. *Let* (X,T) *be a topological space and let* α *be a collection of subsets of* X *having property* (W) *with respect to* T. *If* (X,T) *is separable (*β*-separable = contains a* T-*dense set of cardinality* β*), then* (X,T(α)) *is separable (*β*-separable, respectively).*

Other results about the preservation of separability under extensions of topologies are in Theorem 8 of [4] and Theorem 5.8 of [1].

Recall that a topological space (X,T) is a *Baire Space* provided that the countable intersection of dense open sets is dense. It is easy to see that (X,T) is a Baire space if and only if every T-residual set is T-dense. Therefore, the following result is an immediate consequence of 3.1.

Corollary 3.3. *Let* (X,T) *be a topological space and let* α *be a collection of subsets of* X *having property* (W) *with respect to* T. *Then,* (X,T) *is a Baire space if and only if* (X,T(α)) *is a Baire space.*

Fort [2] has shown that if (X,T) and (X,T*) are weakly equivalent Biare spaces and if a function f from X into a metric space is T*-continuous at each point of a T*-dense set, then f is a T-continuous at each point of a T-dense set. Using this theorem, 2.1, and 3.3, we have the following result:

Theorem 3.4. *Let* (X,T) *be a Baire space and let* α *be a collection of subsets of* X *having property* (W) *with respect to* T. *If a function* f *from* X *into a metric space is* T(α)*-continuous at each point of a* T(α)*-dense set, then* f *is* T*-continuous at each point of a* T*-dense set.*

Let T denote the usual topology on the real line and let $\alpha = \{[a,b): a,b \in R^1\}$. We see that a function $f: R^1 \to R^1$ is right continuous if and only if f is T(α)-continuous (with the usual topology on the range). Hence, by 3.4, we have the classical result in real analysis that if f is right continuous, f is continuous at each point of a dense set.

References

1. C. J. R. Borges, *On extensions of topologies*, Can. J. Math. 19 (1967), 474-487.
2. M. K. Fort, Jr., *Weak equivalence of topologies*, unpublished, University of Georgia (Athens, GA).
3. J. A. Guthrie, D. F. Reynolds, and H. E. Stone, *Connected expansions of topologies*, Bull. Austral. Math. Soc. 9 (1973), 259-265.
4. N. Levine, *Simple extensions of topologies*, Amer. Math. Monthly 71 (1964), 22-25.
5. D. F. Reynolds, *A note on local discrete extensions of topologies*, Kyungpook Math. J. 18 (1978), 91.
6. ________, *Expansions of topologies by collections of closed sets*, Proc. West Virginia Acad. Sci. 47 (1975), 215-218.
7. ________, *Expansions of topologies by locally finite collections*, *Studies in Topology*, Acad. Press, Inc., New York, 1975, N. M. Stavrakas and K. R. Allen, editors, 489-493.

8. ________, *Preservation of connectedness under extensions of topologies*, Kyungpook Math. J. 13 (1973), 217-219.
9. ________ and J. H. Williams, *Simple expansions of topologies*, Proc. West Virginia Acad. Sci. 46 (1974), 177-179.

West Virginia University

Morgantown, West Virginia 26506