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## FEEBLY COMPACT SPACES, MARTIN'S AXIOM, AND "DIAMOND"

by

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## FEEBLY COMPACT SPACES, MARTIN'S AXIOM, AND "DIAMOND"

Jack R. Porter and R. Grant Woods<sup>1</sup>

### 1. Introduction

Must a countably compact, perfect, regular topological space be compact? It has been known for some time that the answer is independent of the usual axioms of set theory. Weiss [We] showed that if one assumes Martin's axiom together with the negation of the continuum hypothesis (henceforth abbreviated  $MA + \neg CH$ ), the answer is "yes," while Ostaszewski [O], assuming  $\diamond$  (a set-theoretic principle following from the axiom of constructibility), constructed a locally compact, zero-dimensional, perfectly normal, countably compact, non-compact, Hausdorff space. (Recall that a space  $X$  is *perfect* if each open set of  $X$  is a union of countably many closed subsets of  $X$ ; it is *perfectly normal* if it is perfect and normal.)

A topological property similar to countable compactness but weaker than it is feeble compactness. A (Hausdorff) space  $X$  is called *feebly compact* if every locally finite family of open subsets of  $X$  is finite. Other characterizations of feeble compactness appear in 2.1. Roughly speaking, feeble compactness is to open sets as countable compactness is to points.

Is it consistent with the usual axioms of set theory that a feebly compact, perfect, regular topological space

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must be compact? The answer is "no"; Isbell's space  $\bar{\Psi}$  is a counterexample. To construct  $\bar{\Psi}$ , choose a maximal almost disjoint family  $\mathcal{M}$  of infinite subsets of  $\mathbb{N}$  (the set of natural numbers). Let  $\bar{\Psi} = \mathbb{N} \cup \mathcal{M}$ , topologized as follows: points of  $\mathbb{N}$  are isolated in  $\bar{\Psi}$ , and if  $M \in \mathcal{M}$ , then  $\{\{M\} \cup A : A \subset \mathbb{N} \text{ and } M \setminus A \text{ is finite}\}$  is a neighborhood base at  $M$ . It is easily verified that  $\bar{\Psi}$  is a locally compact, feebly compact, perfect, but not countably compact, Hausdorff space (see 5I of [GJ] for details).

Suppose we replace "perfect" by a stronger condition in the first sentence of the previous paragraph. Can we obtain an independence result similar to that of Weiss and Ostaszewski? It turns out that we can. Call a space  $X$  *RC-perfect* if each open subset of  $X$  can be written as a union of countably many regular closed subsets of  $X$ . (Recall a subset of  $X$  is *regular closed* if it is the closure of an open set.) The purpose of this paper is to show that if we assume  $MA + \neg CH$ , then every feebly compact, RC-perfect separable regular space is compact; however, if we assume  $\diamond$  there exists a feebly compact, locally compact, RC-perfect zero-dimensional separable Hausdorff space that is not countably compact.

Throughout this paper all hypothesized topological spaces will be assumed to be  $T_3$  (regular Hausdorff).

## 2. "Real" Properties of Feebly Compact Perfect Spaces

We begin by listing some known results and proving some elementary results.

2.1 Proposition. (a) The following are equivalent for a space  $X$ :

(i)  $X$  is feebly compact

(ii) Each infinite family of pairwise disjoint non-empty open sets has a limit point.

(iii) If  $\mathcal{C}$  is a countable collection of open sets with the finite intersection property, then  $\bigcap \{cl_X C : C \in \mathcal{C}\} \neq \emptyset$ .

(iv) If  $\mathcal{C}$  is a countable open cover of  $X$ , there exists a finite subfamily  $\mathcal{J}$  of  $\mathcal{C}$  such that  $X = \bigcup \{cl_X C : C \in \mathcal{J}\}$ .

(b) A Tychonoff space is feebly compact iff it is pseudocompact.

(c) A regular closed subset of a feebly compact space is feebly compact.

The proofs of (a) and (b) can be found in 3.10, 22 and 3.10, 23 of [E]; the proof of (c) is an easy exercise.

2.2 Proposition. Let  $p$  be a point of the feebly compact space  $X$  and let  $\{p\}$  be a  $G_\delta$ -set of  $X$ . Then  $X$  is first countable at  $p$ .

*Proof.* By the regularity of  $X$  and induction on  $\omega$ , one finds a countable sequence  $\{V_n : n \in \omega\}$  of open sets for which  $cl_X V_{n+1} \subset V_n$  for each  $n$ , and  $\{p\} = \bigcap \{V_n : n \in \omega\} = \bigcap \{cl_X V_n : n \in \omega\}$ . If  $W$  is open and  $p \in W$ , find an open set  $T$  for which  $p \in T \subseteq cl_X T \subseteq W$ . Then  $X = T \cup \bigcup \{X \setminus cl_X V_n : n \in \omega\}$ . As  $X$  is feebly compact, by 2.1(a) there exists  $k \in \omega$  for which  $T \cup (X \setminus cl_X V_k)$  is dense in  $X$ . Thus

$X = (cl_X T) \cup (X \setminus V_k)$ , so  $V_k \subseteq cl_X T \subseteq W$ . Thus  $\{V_n : n \in \omega\}$  is a countable neighborhood base at  $p$ .

Recall that a space  $X$  satisfies the *countable chain condition* (written " $X$  is c.c.c.") if  $X$  has no uncountable family of pairwise disjoint non-empty open subsets.

*2.3 Proposition.* *A feebly compact perfect space is c.c.c.*

*Proof.* Our proof follows that used by Juhasz [J] to prove the corresponding result for countably compact spaces (also see [R]). Let  $\{U_\alpha : \alpha < \omega_1\}$  be a collection of  $\aleph_1$  pairwise disjoint non-empty open subsets of  $X$ , and let  $H = cl_X [U\{U_\alpha : \alpha < \omega_1\}] \setminus U\{U_\alpha : \alpha < \omega_1\}$ . Note that  $H \neq \emptyset$  as  $X$  is feebly compact. As  $X$  is perfect, there exists a decreasing countable collection  $\{W_n : n \in \omega\}$  of open subsets of  $X$  such that  $H = \bigcap \{W_n : n \in \omega\}$ . For each  $n \in \omega$ , let  $S_n = \{\alpha < \omega_1 : U_\alpha \setminus cl_X W_n \neq \emptyset\}$ . Then  $S_n$  is finite, for if it were not then  $\{U_\alpha \setminus cl_X W_n : \alpha \in S_n\}$  would be an infinite, locally finite collection of non-empty open subsets of  $X$ , contradicting 2.1(a). Hence there exists  $\delta \in \omega_1 \setminus U\{S_n : n \in \omega\}$ , and  $U_\delta \subseteq \bigcap \{cl_X W_n : n \in \omega\}$ . Choose a non-empty open subset  $V_1$  of  $X$  such that  $cl_X V_1 \subseteq U_\delta$ . Evidently  $V_1 \cap W_1 \neq \emptyset$  and  $V_1 \cap W_1 \cap W_2$  is dense in  $V_1 \cap W_1$ , so using regularity we can choose an open set  $V_2$  such that  $\emptyset \neq cl_X (V_2 \cap W_2) \subseteq V_1 \cap W_1$ . Proceeding inductively in a similar manner, one constructs a decreasing chain  $\{V_k : k \in \omega\}$  of open sets of  $X$  for which  $\emptyset \neq cl_X (V_{k+1} \cap W_{k+1}) \subseteq V_k \cap W_k$  for each  $k \in \omega$ . Now by 2.1(a)  $\bigcap \{cl_X (V_n \cap W_n) : n \in \omega\} \neq \emptyset$ . But  $\bigcap \{cl_X (V_n \cap W_n) : n \in \omega\} \subseteq H \cap U_\delta = \emptyset$ , which is a contradiction. The proposition follows.

2.4 *Corollary.* A feebly compact, perfect space is c.c.c. and first countable.

**3. Feebly Compact RC-Perfect Spaces and Martin's Axiom**

The following set-theoretic principle, denoted  $P(c)$ , is known to be a consequence of Martin's axiom (see [K]).

$P(c)$  is the statement:

If  $\kappa < 2^{\aleph_0}$  and if  $\{A(\alpha) : \alpha < \kappa\}$  is a collection of  $\kappa$  subsets of  $\omega$  for which  $\bigcap \{A(\alpha) : \alpha \in F\}$  is infinite whenever  $F$  is a finite subset of  $\kappa$ , then there exists an infinite subset  $B$  of  $\omega$  such that  $B \setminus A(\alpha)$  is finite for each  $\alpha \in \kappa$ .

The proof of the following lemma is essentially the same as that of Hechler's corresponding result for countably compact, separable spaces (see [H]). Recall that a space  $X$  has *countable  $\pi$ -weight* if there is a countable collection  $\mathcal{C}$  of non-empty open subsets of  $X$  such that if  $V$  is a non-empty open subset of  $X$ , then there exists  $C \in \mathcal{C}$  such that  $C \subseteq V$ . We denote  $2^{\aleph_0}$  by  $c$ .

3.1 *Lemma.* Assume  $P(c)$ . If  $X$  is a feebly compact space of countable  $\pi$ -weight, and if  $\mathcal{U}$  is an open cover of  $X$  for which  $|\mathcal{U}| < c$ , then there exists a finite subcollection  $\mathcal{J}$  of  $\mathcal{U}$  such that  $X = \bigcup \{\text{cl}_X F : F \in \mathcal{J}\}$ .

*Proof.* Let  $\{V_n : n \in \omega\}$  be a faithfully indexed countable  $\pi$ -base, and let  $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ , where  $\kappa < c$ . For each  $\alpha < \kappa$  let  $A(\alpha) = \{n \in \omega : V_n \cap U_\alpha \neq \emptyset\}$ . Suppose that  $F$  is a finite subset of  $\kappa$  for which  $\bigcup \{A(\alpha) : \alpha \in F\} = \omega$ . If  $W$  is a non-empty open subset of  $X$ , find  $m \in \omega$  such that  $V_m \subset W$ . There exists  $\alpha(m) \in F$  for which  $m \in A(\alpha(m))$ , and

so  $V_m \cap U_{\alpha(m)} \neq \phi$ . Thus  $W \cap U\{U_\alpha : \alpha \in F\} \neq \phi$ , so  $\{U_\alpha : \alpha \in F\}$  is our  $\mathcal{J}$ .

Suppose  $F$  were a finite subset of  $\kappa$  and  $\omega \setminus U\{A(\alpha) : \alpha \in F\} = G$  were finite. If  $m \in G$  there exists  $\alpha(m) < \kappa$  such that  $V_m \cap U_{\alpha(m)} \neq \phi$ . Let  $H = \{\alpha(m) : m \in G\}$ ; then  $U\{A(\alpha) : \alpha \in F \cup H\} = \omega$ , and argue as above.

So, suppose that  $\omega \setminus U\{A(\alpha) : \alpha \in F\} = \omega \setminus A(\alpha) : \alpha \in F$  is infinite for every finite subset  $F$  of  $\kappa$ . By P(c) there exists an infinite subset  $S$  of  $\omega$  such that  $S \cap A(\alpha)$  is finite for each  $\alpha < \kappa$ . As  $X$  is feebly compact,  $\{V_n : n \in S\}$  has a cluster point  $p$ . As  $\mathcal{U}$  covers  $X$ , choose  $\delta < \kappa$  such that  $p \in U_\delta$ . Then  $S \cap A(\delta)$  must be infinite, which is a contradiction.

It is not possible in 3.1 to replace the condition that  $X$  have countable  $\pi$ -weight by the condition that  $X$  be separable. This is illustrated by example 3.3 below. First we introduce a useful concept. Recall (see 17K of [Wi]) that a Hausdorff space is *H-closed* if it is closed in any Hausdorff space in which it is embedded. We will use the following two well-known properties of H-closed spaces; see 17K and 17L of [Wi].

3.2 Lemma. (a) *Each H-closed regular space is compact.*

(b) *A Hausdorff space  $X$  is H-closed iff given an open cover  $\mathcal{C}$  of  $X$ , there exists a finite subfamily  $\mathcal{J}$  of  $\mathcal{C}$  such that  $X = \bigcup\{cl_X F : F \in \mathcal{J}\}$ .*

3.3 Example. Assume  $\neg CH$ . Let  $\underline{2}$  denote the two-point discrete space, let  $p$  be a fixed point of  $\underline{2}^{\omega_1}$ , and let

$X = \underline{2}^{\omega_1} \setminus \{p\}$ . Then  $X$  is separable (see 16.4 of  $[W_1]$ ), Tychonoff, non-compact, and has weight  $\aleph_1$ , which is less than  $c$ . It follows from 3.2 that  $X$  has an open cover  $\mathcal{U}$  with  $|\mathcal{U}| < c$  such that  $\cup\{cl_X F : F \in \mathcal{F}\} \neq X$  for each finite subfamily  $\mathcal{F}$  of  $\mathcal{U}$ . There obviously exists a point  $q \in \underline{2}^{\omega_1}$  such that the  $\Sigma$ -product  $\Sigma(q)$  based at  $q$  does not contain  $p$  (see 2.7.13, page 158 of  $[E]$ ). Thus  $\Sigma(q) \subseteq X \subseteq \underline{2}^{\omega_1}$ . By 3.12.23 on page 305 of  $[E]$   $\beta(\Sigma(q)) = \underline{2}^{\omega_1}$ , so  $\beta X = \underline{2}^{\omega_1}$  (see 6.7 of  $[GJ]$ ). It follows from 6J of  $[GJ]$  that  $X$  is pseudocompact and thus feebly compact. Hence  $X$  witnesses the failure of 3.1 if "countable  $\pi$ -weight" is replaced by "separable" in a model of set theory in which CH fails.

Recall our blanket assumption (implicitly lifted while we defined H-closed spaces) that all hypothesized spaces are regular.

*3.4 Theorem.* *Assume  $P(c)$  and the negation of the continuum hypothesis. If  $X$  is a separable, RC-perfect, feebly compact space, then  $X$  is compact.*

*Proof.* By 2.1(a) and 3.2(b) a Lindelöf feebly compact space is H-closed and by 3.2(a) an H-closed regular space is compact. Hence it suffices to show that  $X$  is Lindelöf.

So, suppose  $X$  is not Lindelöf and that  $\mathcal{U}$  is an open cover of  $X$  with no countable subcover. We inductively choose points  $\{x_\alpha : \alpha < \omega_1\}$ , open sets  $\{G_\alpha : \alpha < \omega_1\}$ , and a subfamily  $\{U_\alpha : \alpha < \omega_1\} \subseteq \mathcal{U}$  as follows. Let  $\alpha < \omega_1$  and assume we have chosen points  $\{x_\delta : \delta < \alpha\}$ , open sets  $\{G_\delta : \delta < \alpha\}$ , and a subfamily  $\{U_\delta : \delta < \alpha\}$  of  $\mathcal{U}$  satisfying these conditions:



- (i)  $x_\delta \in G_\delta \subseteq \text{cl}_X G_\delta \subseteq U_\delta$  for each  $\delta < \alpha$   
(ii)  $x_\delta \in U_\delta \setminus \{U_\gamma : \gamma < \delta\}$  for each  $\delta < \alpha$ .

As  $\mathcal{U}$  has no countable subcover, we can choose  $x_\alpha \in X \setminus \{U_\delta : \delta < \alpha\}$ . As  $\mathcal{U}$  covers  $X$  find  $U_\alpha \in \mathcal{U}$  such that  $x_\alpha \in U_\alpha$ . Choose  $G_\alpha$  using the regularity of  $X$ . Now (i) and (ii) are satisfied for each  $\alpha < \omega_1$ .

Let  $G = \{G_\alpha : \alpha < \omega_1\}$ . As  $X$  is RC-perfect, there exist countably many regular closed subsets of  $X$ --say  $\{A_n : n \in \omega\}$ --such that  $G = \{A_n : n \in \omega\}$ . By 2.1(c) each  $A_n$  is feebly compact. As  $X$  is separable, so is each  $\text{int}_X A_n$  and hence each  $A_n$ . By 2.4  $X$  is first countable, so each  $A_n$  is too. A separable first countable space has countable  $\pi$ -weight, so each  $A_n$  has countable  $\pi$ -weight. Thus by 3.1 there is a finite subset  $F_n$  of  $\omega_1$  such that  $A_n \subseteq \{ \text{cl}_X (G_\alpha \cap A_n) : \alpha \in F_n \}$ . Let  $F = \{F_n : n \in \omega\}$ . Then  $G = \{ \text{cl}_X (G_\alpha \cap A_n) : \alpha \in F_n \text{ and } n \in \omega \} \subseteq \{ \text{cl}_X G_\alpha : \alpha \in F \} \subseteq \{U_\alpha : \alpha \in F\}$ . As  $F$  is countable, choose  $\beta < \omega_1$  so that  $\beta > \alpha$  for each  $\alpha \in F$ . Then  $x_\beta \in G \setminus \{U_\alpha : \alpha \in F\}$ , which is a contradiction. Thus  $X$  is Lindelöf and hence compact.

It is consistent with the usual axioms of set theory that there exist a feebly compact RC-perfect space that is not separable--a compact Souslin line is such a space (see [K]). From 2.4 we know that a feebly compact RC-perfect space must always be c.c.c. and first countable. It has been proved by Juhász [J] that  $\text{MA} + \neg \text{CH}$  implies that a compact first countable c.c.c. space must be hereditarily separable (a proof of this is given in [R]). This suggests the following question, which we have been unable to resolve.

3.5 *Question.* Assume  $MA + \neg CH$ . Must a feebly compact RC-perfect space necessarily be separable?

There are a number of supplementary conditions on a feebly compact RC-perfect space that will guarantee that it be separable. We mention two. Recall that if  $\lambda$  is a cardinal, then a Tychonoff space  $X$  is an *absolute*  $G_\lambda$  if it can be written as the intersection of no more than  $\lambda$  open subsets of some compactification of  $X$  (equivalently, all compactifications of  $X$ ). We let  $L(X)$  denote the locally compact part of the Tychonoff space  $X$ --i.e.  $L(X) = \{p \in X: p \text{ has a compact neighborhood}\}$ .

3.6 *Theorem.* Assume  $MA + \neg CH$ . Let  $X$  be a feebly compact RC-perfect space. If either:

- (a)  $X$  is an absolute  $G_\lambda$  for some  $\lambda < c$ , or
- (b)  $L(X)$  is dense in  $X$ ,

then  $X$  is compact.

*Proof.* (a) In 4.5 of [T], Tall proves that if  $X$  is an absolute  $G_\lambda$  (for  $\lambda < c$ ), and  $X$  is first countable and c.c.c., then  $MA + \neg CH$  implies that  $X$  is separable. Our result then follows from 2.4 and 3.4.

(b) Let  $\mathcal{M}$  be a maximal family of pairwise disjoint compact regular closed subsets of  $X$ . By 2.4  $X$  is c.c.c. and first countable, so each member of  $\mathcal{M}$  is also. As noted above, it follows from a theorem of Juhász that  $MA + \neg CH$  implies that a compact, first countable, c.c.c. space is separable. Hence each member of  $\mathcal{M}$  is separable. As  $X$  is c.c.c.,  $|\mathcal{M}| \leq \aleph_0$  so  $\cup\{M: M \in \mathcal{M}\}$  is separable. As  $L(X)$  is

dense in  $X$ , so is  $\cup\{M: M \in \mathcal{M}\}$ . Hence  $X$  is separable (and therefore compact).

#### 4. An Example Using $\diamond$

Suppose that it followed from the usual axioms of set theory that a separable feebly compact RC-perfect space were countably compact. Then 3.4 would follow immediately from Weiss's result [We] that assuming  $P(c)$ , each countably compact regular perfect space is compact. We show that this line of argument cannot be used by producing (assuming  $\diamond$ ) a feebly compact, RC-perfect, locally compact Hausdorff space that has a countable dense set of isolated points but which is not countably compact.

Throughout this section  $\{\lambda_\alpha: \alpha < \omega_1\}$  will denote an order-preserving indexing of the countable limit ordinals. Our procedure is a modification of Ostaszewski's well-known construction (see [O]). Roughly speaking, we topologize  $\omega_1$  inductively so that  $\omega$  forms a countable dense set of isolated points. We assign limit points to subsets of  $\omega_1$  so that every infinite subset of  $\omega$  gets a limit point (thus making our space pseudocompact), but so that  $\{\lambda_\alpha: \alpha < \omega\}$  receives no limit point (thus preventing our space from being countably compact).

Recall (see [R]) that the set-theoretic principle  $\diamond$  is equivalent to the continuum hypothesis together with the set-theoretic principle  $\clubsuit$ . Recall that  $\clubsuit$  says:

There is a collection  $\{T_\alpha: \alpha < \omega_1\}$  of  $\omega_1$  countable subsets of  $\omega_1$  such that for each  $\alpha < \omega_1$ ,  $T_\alpha$  is a cofinal subset of  $\lambda_\alpha$ , and such that if  $H$  is an uncountable subset

of  $\omega_1$ , then there exists  $\alpha < \omega_1$  for which  $T_\alpha \subseteq H$ .

A collection of subsets of  $\omega_1$  with the above properties will be said to *witness*  $\clubsuit$ . As usual, an ordinal  $\delta$  is viewed as being the set of ordinals less than  $\delta$ . To emphasize this, sometimes we write  $[0, \delta)$  in place of  $\delta$ , and  $[0, \delta]$  in place of  $[0, \delta+1]$ , and so on.

4.1 Lemma. Assume  $\clubsuit$ . Then there exists a collection  $\{S_\alpha : \alpha < \omega_1\}$  of countable subsets of  $\omega_1$  that witnesses  $\clubsuit$  and for which  $[U\{S_\alpha : \alpha < \omega_1\}] \cap \{\lambda_\delta : \delta \leq \omega\} = \emptyset$ .

Proof. Let  $\{T_\alpha : \alpha < \omega_1\}$  witness  $\clubsuit$ . Define  $\{S_\alpha : \alpha < \omega_1\}$  as follows:

(i)  $S_0 = T_0$

(ii) If  $0 < n < \omega$  let  $S_n = T_n \setminus \{\lambda_j : j < n\}$

(iii)  $S_\omega = \{S_n : n < \omega\}$ .

(iv) If  $\alpha > \omega$  let  $S_\alpha = T_\alpha \setminus (0, \lambda_\omega + 1)$

Obviously  $[U\{S_\alpha : \alpha < \omega_1\}] \cap \{\lambda_\delta : \delta \leq \omega\} = \emptyset$  and one easily verifies that  $\{S_\alpha : \alpha < \omega_1\}$  witnesses  $\clubsuit$ .

4.2 Theorem. Assume  $\diamond$ . There exists a space  $X$  with the following properties:

(1)  $|X| = \aleph_1$

(2)  $X$  is a locally countable, locally compact zero-dimensional Hausdorff space.

(3)  $X$  has a countable dense set of isolated points.

(4) Each open subset of  $X$  is either countable or has a countable complement.

(5)  $X$  is pseudocompact but not countably compact.

(6)  $X$  is RC-perfect.

*Proof.* Let  $\{X_\alpha : \alpha < \omega_1\}$  be an indexing of the infinite subsets of  $\omega$  ( $\omega$ , not  $\omega_1$ ). Such an indexing exists as the continuum hypothesis holds. Let  $\{S_\alpha : \alpha < \omega_1\}$  be as constructed in 4.1. Let  $\{\mathbb{N}_i : i \in \omega\}$  be a partition of  $\omega$  into countably many infinite subsets. Our construction closely mimics that used in [O] and described in [R].

For each  $\beta < \omega_1$  and each  $n < \omega$  we will define a subset  $U_{\beta,n}$  of  $\omega_1$ . If  $\beta < \omega$  set  $U_{\beta,n} = \{\beta\}$  for each  $n \in \omega$ . Our construction now proceeds inductively as follows. Let  $0 < \gamma < \omega_1$  and assume for each  $\alpha < \gamma$  and each  $\beta < \lambda_\alpha$  we have defined  $U_{\beta,n}$  so that the following are satisfied:

(i)  $\{U_{\beta,n} : n < \omega \text{ and } \beta < \lambda_\alpha\}$  is an open base for a locally compact zero-dimensional Hausdorff topology  $\tau_\alpha$  on  $[0, \lambda_\alpha)$ , and if  $\delta < \lambda_\alpha$  then  $[0, \delta)$  is open in  $[0, \lambda_\alpha)$ .

(ii)  $\{U_{\beta,n} : n < \omega\}$  is a decreasing sequence of compact open subsets of  $([0, \lambda_\alpha), \tau_\alpha)$  and forms a neighborhood base at  $\beta$  in this space.

(iii) If  $\beta < \lambda_\alpha$  and  $n < \omega$  then  $U_{\beta,n} \subseteq [0, \beta]$ .

(iv) If  $\beta < \lambda_\alpha$  and  $\beta \notin \{\lambda_i : i < \omega\}$  then  $U_{\beta,n} \cap \{\lambda_i : i < \omega\} = \emptyset$  for each  $n \in \omega$ .

(v) If  $0 < \delta < \alpha$  then there exists an increasing, cofinal subsequence  $J_\delta = \{j_{\delta,n} : n \in \omega\}$  of  $S_\delta$  such that  $J_\delta \cap [0, \omega] = \emptyset$  and  $U_{\lambda_\delta+i,n}$  contains an infinite subset of  $J_\delta$  for each  $i \in \omega$  and each  $n \in \omega$ .

(vi) If  $\beta < \alpha$  then  $X_\beta$  has a limit point in  $[0, \lambda_\alpha)$ .

Now we will define  $\{U_{\beta,n} : \beta < \lambda_\gamma \text{ and } n \in \omega\}$  so that (i)-(v) above are satisfied when  $\gamma$  is replaced by  $\gamma + 1$ . First note that if  $\gamma$  is a limit ordinal, then we have already

done this. So, assume  $\gamma = \alpha + 1$  for some  $\alpha$ . Then  $\lambda_\gamma = \lambda_\alpha + \omega$ , and we have defined  $U_{\beta,n}$  for each  $\beta < \lambda_\alpha$  and each  $n < \omega$ . It remains to define  $U_{\beta,n}$  for each  $\beta$  of the form  $\lambda_\alpha + i$  (where  $i < \omega$ ), and each  $n \in \omega$ .

If  $X_\alpha$  has no limit point in  $([0, \lambda_\alpha), \tau_\alpha)$ , let  $X_\alpha = \{a_{\alpha,n} : n \in \omega\}$ . Choose  $J_\alpha \subset S_\alpha \setminus [0, \omega]$  such that  $J_\alpha = \{j_{\alpha,n} : n \in \omega\}$ ,  $j_{\alpha,n} < j_{\alpha,n+1}$  for each  $n \in \omega$ , and  $J_\alpha$  is cofinal in  $[0, \lambda_\alpha)$ . Note that  $J_\alpha \cup X_\alpha$  is a closed, countably infinite discrete subset of  $[0, \lambda_\alpha)$  and  $J_\alpha \cap X_\alpha = \emptyset$ . Thus for each  $n \in \omega$  there exists  $k(n) \in \omega$  such that  $U_{j_{\alpha,n}, k(n)} \cap (J_\alpha \cup X_\alpha) = \{j_{\alpha,n}\}$ . Let  $W_n = U_{j_{\alpha,n}, k(n)}$ . For each  $i \in \omega$  and  $n \in \omega$  set  $U_{\lambda_\alpha+i, n} = \{\lambda_\alpha + i\} \cup \{a_{\alpha,k} : k \in \mathbb{N}_i \text{ and } k > n\} \cup \{U\{W_k \setminus U\{W_i : i < k\} : k > n \text{ and } k \in \mathbb{N}_i\}$ .

If  $X_\alpha$  has a limit point in  $([0, \lambda_\alpha), \tau_\alpha)$ , choose  $W_k$  as above, except that we no longer require that  $|W_k \cap X_\alpha| \leq 1$ . Let  $U_{\lambda_\alpha+i, n} = \{\lambda_\alpha + i\} \cup \{U\{W_k \setminus U\{W_i : i < k\} : k \in \mathbb{N}_i \text{ and } k > n\}$ .

It is now tedious but straightforward to check that (i)-(v) are satisfied when  $\gamma$  is replaced by  $\gamma + 1$ .

We topologize  $\omega_1$  by letting  $\{U_{\beta,n} : \beta < \omega_1 \text{ and } n < \omega\}$  serve as an open base for a topology  $\tau$  on  $\omega_1$ . From (i) it follows that the resulting space  $X$  is a locally compact, locally countable zero-dimensional Hausdorff space, so (1) and (2) are satisfied.

We now assert the following for each  $\alpha < \omega_1$ :

If  $\beta \geq \lambda_\alpha$  then  $\beta \in \text{cl}_X J_\alpha$ . --- (\*)

We verify this by inducting on  $[\lambda_\alpha, \omega_1)$ . It follows from (ii) and (v) that  $\lambda_\alpha + i \in \text{cl}_X J_\alpha$  for each  $i < \omega$ . Suppose

(\*) is false, and let  $\beta$  be the smallest member of  $[\lambda_\alpha, \omega_1)$  that is not in  $\text{cl}_X J_\alpha$ . Thus  $\beta \geq \lambda_{\alpha+1}$ . Thus there exists some  $\delta \geq \alpha + 1$  such that  $U_{\beta, n}$  contains a cofinal subset of  $J_\delta$ . Thus  $U_{\beta, n} \cap [\lambda_\alpha, \beta) \neq \emptyset$  for each  $n \in \omega$ . By the minimality of  $\beta$  it follows that  $U_{\beta, n} \cap J_\alpha \neq \emptyset$  for each  $n \in \omega$ . Thus  $\beta \in \text{cl}_X J_\alpha$ , which is a contradiction. Thus (\*) is true.

It now follows that  $\omega$  forms a countable dense set of isolated points of  $X$ . Each  $\beta \in \omega$  is isolated as  $U_{\beta, n} = \{\beta\}$ . Set  $\alpha = 0$  in (\*) and note that  $J_0 \subseteq \omega$  and  $\lambda_0 = \omega$ . Thus (3) holds.

If  $\{V_n : n \in \omega\}$  is a countably infinite family of pairwise disjoint non-empty open sets of  $X$ , choose  $j_n \in \omega \cap V_n$  for each  $n$ . Then  $\{j_n : n \in \omega\} = X_\alpha$  for some  $\alpha < \omega_1$ , and hence has a limit point in  $X$  by virtue of our inductive construction. This is a cluster point of  $\{V_n : n \in \omega\}$ , so by 2.1  $X$  is pseudocompact.

It follows from (iv) that  $\{\lambda_\delta : \delta \leq \omega\}$  is a countably infinite subset of  $X$  with no limit point, so  $X$  is not countably compact. Thus (5) is verified.

It follows by the same reasoning as used in Ostaszewski's original construction that each open subset of  $X$  is either countable or co-countable. For if  $V$  is open and  $X \setminus V$  is not countable, then as  $\{S_\alpha : \alpha < \omega_1\}$  witnesses  $\clubsuit$  there exists  $\alpha < \omega_1$  for which  $S_\alpha \subseteq \omega_1 \setminus V$ . By (\*) above it follows that  $[\lambda_\alpha, \omega_1) \subseteq \text{cl}_X J_\alpha \subseteq \text{cl}_X S_\alpha \subseteq \omega_1 \setminus V$ , so  $V$  is countable. Thus (4) holds.

To show that  $X$  is RC-perfect, suppose that  $V$  is open in  $X$ . If  $V$  is countable, then by the regularity of  $X$  there

exists, for each  $p \in V$ , a regular closed subset  $A(p)$  of  $X$  for which  $p \in \text{int}_X A(p) \subseteq A(p) \subseteq V$ . Thus  $V = \cup\{A(p) : p \in V\}$  and  $V$  is a union of countably many regular closed sets.

If  $V$  is not countable, by (4) there exists  $\alpha < \omega_1$  such that  $X \setminus V \subseteq [0, \lambda_\alpha)$ . As  $\{S_\alpha : \alpha < \omega_1\}$  witnesses  $\clubsuit$ , find  $\delta < \omega_1$  such that  $S_\delta \subseteq [\lambda_\alpha, \omega_1)$ . Thus  $S_\delta \subseteq V \cap [0, \lambda_\delta)$ . Now  $[0, \lambda_\delta)$  is a countable subspace of  $X$  and is thus paracompact. Evidently  $J_\delta$  is a closed discrete subspace of  $[0, \lambda_\delta)$  and  $J_\delta \subseteq V \cap [0, \lambda_\delta)$ . There exist a pairwise disjoint countably infinite family  $\{B_n : n \in \omega\}$  of compact open subsets of  $[0, \lambda_\delta)$  such that  $J_\delta \subseteq \cup\{B_n : n \in \omega\} \subseteq V \cap [0, \lambda_\delta)$ . The countable open cover  $\{[0, \lambda_\delta) \setminus J_\delta\} \cup \{B_n : n \in \omega\}$  of  $[0, \lambda_\delta)$  has a precise locally finite open refinement  $\mathcal{W}$ . For each  $n \in \omega$  find a regular closed subset  $A_n$  of  $[0, \lambda_\delta)$  and  $W_n \in \mathcal{W}$  such that  $A_n \subseteq W_n \subseteq B_n$  and  $J_\delta \subseteq \cup\{\text{int}_{[0, \lambda_\delta)} A_n : n \in \omega\}$ . As  $\{A_n : n \in \omega\}$  is a locally finite family in  $[0, \lambda_\delta)$ , we see that  $\cup\{A_n : n \in \omega\}$  is a regular closed subset of  $[0, \lambda_\delta)$  and contained in  $V$ . Now  $J_\delta \subseteq \cup\{A_n : n \in \omega\}$  and  $[\lambda_\delta, \omega_1) \subseteq \text{cl}_X J_\delta$  by (\*). As  $[0, \lambda_\delta)$  is open in  $X$ , so is  $\text{int}_{[0, \lambda_\delta)} A_n$ . Thus  $\text{cl}_X [\cup\{\text{int}_{[0, \lambda_\delta)} A_n : n \in \omega\}] = \cup\{A_n : n \in \omega\} \cup [\lambda_\delta, \omega_1) \subseteq V$ . Thus we have produced a regular closed subset of  $X$  contained in  $V$  and containing all but countably many points of  $V$ . Arguing as in the case where  $V$  was countable, we now conclude that  $V$  is the union of countably many regular closed subsets of  $X$  and (6) follows.

4.3 Remark. The example in 4.2 is pseudocompact and not countably compact, and hence neither normal nor countably paracompact. In theorems 3 and 4 of [A], Aull proves that if  $X$  is regular, and if each closed subset  $F$  of  $X$  is the



intersection of countably many regular closed sets each of which contains  $F$  in its interior, then  $X$  is normal and countably paracompact. The example in 4.2 shows that we cannot weaken the hypothesis to require only that each  $F$  be the intersection of countably many regular open subsets of  $X$  (but not necessarily the intersection of their closures).

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