Research Announcement:
CONTINUA OF CONSTANT DISTANCES RELATED TO THE SPANS

by
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The purpose of this note is to announce some results which complement and can, perhaps, offer a better handling of the concept of the span for compact metric spaces. A complete version with proofs and applications will be published elsewhere.

All spaces are assumed to be non-empty metric spaces, and all mappings to be continuous functions. Let \( f : X \to Y \) be a mapping. If \( X \) is connected, the surjective span \( \sigma^*(f) \) of \( f \) is defined to be the least upper bound of the set of real numbers \( \alpha \) with the following property: there exist non-empty connected sets \( C_\alpha \subset X \times X \) such that \( \text{dist}[f(x), f(x')] \geq \alpha \) for \( (x, x') \in C_\alpha \), and

\[
\begin{align*}
\sigma^* \quad & p_1(C_\alpha) = p_2(C_\alpha) = X, \\
(\sigma^*) \quad & p_1(C_\alpha) = p_2(C_\alpha) = X,
\end{align*}
\]

where \( p_1 \) and \( p_2 \) denote the standard projections of the product, that is, \( p_1(x, x') = x \) and \( p_2(x, x') = x' \). The span \( \sigma(f) \), the semispan \( \sigma_0(f) \), both for mappings \( f \) with the domains \( X \) not necessarily connected, and the surjective semispan \( \sigma^*_0(f) \) in the case of connected domains, are defined similarly with condition \( (\sigma^*) \) relaxed to conditions

\[
\begin{align*}
\sigma \quad & p_1(C_\alpha) = p_2(C_\alpha), \\
(\sigma_0) \quad & p_1(C_\alpha) \supseteq p_2(C_\alpha), \\
(\sigma^*_0) \quad & p_1(C_\alpha) = X,
\end{align*}
\]

respectively. The following inequalities are direct consequences of the definitions:
\[0 \leq \sigma^*(f) \leq \sigma(f) \leq \sigma_0(f) \leq \text{diam } Y,\]
\[0 \leq \sigma^*(f) \leq \sigma^*_0(f) \leq \sigma_0(f) \leq \text{diam } Y.\]

For \( \tau = \sigma, \sigma^*, \sigma_0, \sigma^*_0 \), the corresponding spans \( \tau(X) \) of a space \( X \) are the spans \( \tau(\text{id}_X) \) of the identity mapping on \( X \).

The span \( \sigma(f) \) of a mapping \( f \) was originally defined by Ingram [1], while the present author earlier introduced the span \( \sigma(X) \) and, subsequently, the other types of these quantities for metric spaces (see [2] and [3]). It is known that, for some particular spaces, neither two of these four types of spans need to be equal (see [3] and [4]).

One of the consequences of our results is that, in the definition of the span \( \sigma(f) \) for compact domains, the inequality \( \text{dist}[f(x), f(x')] \geq \alpha \) can be replaced by the equality. The author wishes to thank M. B. de Castro and L. G. Oversteegen for helpful discussions concerning such a possibility. It can be derived directly from the theorem below, and the same replacement can also be made in the definitions of other types of spans. The two kinds of definitions, one using the equality and another one using the inequality, are then equivalent, respectively, in each of the four cases of the concepts involved for mappings of compact metric spaces. By a continuum we understand a connected compact metric space.

**Theorem.** If \( f : X \to Y \) is a mapping, \( X \) is a compact metric space, \( \tau = \sigma, \sigma_0 \), and \( 0 \leq \beta \leq \tau(f) \), then there exists a non-empty continuum \( K_\beta \subset X \times X \) such that
\[
\text{dist}[f(x), f(x')] = \beta
\]
for \((x, x') \in K_\beta\), and condition \((\tau)\) is satisfied for \(K_\beta\) in lieu of \(C_\alpha\), respectively. Moreover, if \(X\) is a continuum, the same conclusion also holds for \(\tau = \sigma^*, \sigma_0^*\).

The scheme of the proof is illustrated, rather vaguely, in the following diagram:

The space \(X\) is considered embedded in the Hilbert space \(R^\omega\), and its image \(f(X)\) is considered embedded in the Hilbert cube \(I^\omega\). The mapping \(f\) is extended over \(R^\omega\) to a mapping \(\bar{f}\). For each \(n = 1, 2, \ldots\), a polyhedron \(P_n\) is taken in the \((1/n)\)-neighborhood of \(X\) in \(R^\omega\) such that there exists a \((1/n)\)-translation \(g_n\) of \(X\) into \(P_n\). The space \(U_n\) is the universal covering space of \(P_n\) with \(h_n\) the covering projection. We use the unicoherence of the product \(U_n \times U_n\) to find a connected set \(M_n\) which cuts \(U_n \times U_n\) between certain two points and is contained in a properly selected subset of \(U_n \times U_n\). The desired continuum \(K_\beta\) is then defined to be the limit of a convergent subsequence of the sequence of the closures of the sets \((h_n \times h_n)(M_n)\) \((n = 1, 2, \ldots)\).
References


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