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COMPACTIFICATIONS OF THE RAY WITH THE CLOSED ARC AS REMAINDER

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0. Introduction and Summary

Let J denote the ray (0,1] and let I denote the closed interval [0,1]. If f: J → I is a continuous function, let Jf denote the graph of f. Let $\alpha_f J$ denote the closure, $\overline{Jf}$, of Jf in I × I. Then $\alpha_f J$ is a compactification of J, since the function h: J → $\alpha_f J$ given by h(t) = (t,f(t)) is a dense embedding of J into $\alpha_f J$. Let $\hat{J}f$ denote the remainder $\alpha_f J \backslash Jf$. In (1) a procedure similar to the above is used to obtain compactifications for a large class of non-compact, locally compact spaces. Techniques using the closure of the graph of a function f are used in (2) to obtain various topological extensions of f.

It is readily seen that if f: J → I is continuous and is continuously extendible to I, then $\alpha_f J$ is homeomorphic (≅) to I (the one point compactification of J). Let F denote the class of all functions f: J → I, which are continuous but are not continuously extendible to I. If f ∈ F, then $\hat{J}f$ is a closed subinterval of I and $\alpha_f J$ is non-locally connected because Jf is forced to oscillate as it approaches $\hat{J}f$. In (3) and (4), the author and S. Khabbaz develop invariants to study the homeomorphism and the

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homotopy types of the spaces $Jf \cup \{(0,0)\}$, where $f \in F$.

Our purpose here is to study the compactifications $\alpha_f J$, where $f \in F$. For related work see (5) and (6).

In Section 1 we associate with each $\alpha_f J$, $f \in F$, a closed ordered subset Ef of $\hat{Jf}$, where the order is that induced by the natural order on $\hat{Jf}$. Ef is called the type of the compactification $\alpha_f J$ and consists of those points of $\hat{Jf}$ arbitrarily close to which Jf makes significant turns. In theorem 1.4, Ef is proved to be a topological invariant of $\alpha_f J$. In Section 2 we prove a reduction theorem that associates with each $\alpha_f J$, another compactification $\alpha_g J$, homeomorphic to $\alpha_f J$, where g is piecewise linear over a sequence V in J converging to 0, and where each $v \in V$, is a local extremum of g. Moreover $\alpha_g J$ has the nice property that $Eg = \overline{Vg}\setminus Vg$, where $Vg = \{(v,g(v)): v \in V\}$. Hence Vg enjoys some sort of minimality in the sense that it contains no subsequences converging to any point of $\hat{Jg}\setminus Eg$.

Finally, in Section 3, we prove that for each closed subset T of I, there exists continuum many nonhomeomorphic compactifications of the ray, all of which have type T.

1. The Invariant Ef

Definition 1.1. Let p,q be two points in Jf. Then $[p,q]_f$ denotes the closed arc in Jf joining p and q. $[p,q]_f$ is called a *wedge* (respectively a *spike*) if the lowest (highest) points of $[p,q]_f$ are all interior points. Such a wedge or spike is called *symmetric* if $\pi_2(p) = \pi_2(q)$, where π_2 is the projection onto the y-coordinate. Finally if p, q $\in$ I $\times$ I, then [p,q] denotes the straight line segment in I $\times$ I joining p and q.

Definition 1.2. [See (7) or (8)]. Let $\{A_i\}$ be a sequence of nonempty closed subsets of $\alpha_f J$. Then define:

(a) Lim $\inf\{A_i\} = \{x \in \alpha_f J$: if U is an open neighborhood of x in $\alpha_f J$, then $U \cap A_i \neq \phi$ for all but finitely many i}.

(b) Lim $\sup\{A_i\} = \{x \in \alpha_f J$: if U is an open neighborhood of x in $\alpha_f J$, then $U \cap A_i \neq \phi$ for infinitely many i}. If $\lim \inf\{A_i\} = A = \lim \sup\{A_i\}$, then we say that the sequence $\{A_i\}$ converges to A, or $\lim \{A_i\} = A$.

The above definition of convergence is equivalent to convergence with respect to the Hausdorf metric on the set of all nonempty closed subsets of $\alpha_f J$. See for example (8).

Definition 1.3. Let s $\in \hat{J}f$. Then s is called *essential* in $\alpha_f J$, if it satisfies one of the following two conditions:

(i) There exists a sequence $\{[p_i,q_i]_f\}$ of wedges (spikes) in Jf and a positive number ε, such that $\lim\{[p_i,q_i]_f\} = [s,s+\varepsilon]([s,s-\varepsilon])$, and $\lim\{p_i\} = \lim\{q_i\} = s + \varepsilon(s - \varepsilon)$.

(ii) s is the limit of a sequence of points in $\hat{J}f$ satisfying condition (i). Otherwise s is called *inessential* in $\alpha_f J$. *Let* Ef *denote the set of essential points of* $\alpha_f J$, *ordered by the natural order on* Jf. (In figure 1 Ef = $\{0,\frac{1}{3},1\}$.)

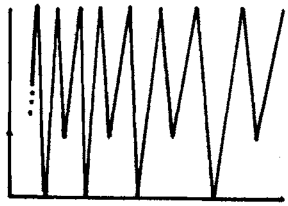

Figure 1

Theorem 1.4. *Let* $f, g \in F$ *and let* $h\colon \alpha_f J \to \alpha_g J$ *be a homeomorphism. Then* $h|Ef$ *is a monotone homeomorphism onto* Eg.

The proof of this theorem follows from Lemma 1.6. First we need the following:

Convention 1.5.

(i) If a statement P is made about the elements of a sequence S, such that all but finitely many elements of S satisfy P, then we say S *almost satisfies* P, or almost each element of S satisfies P.

(ii) Let $X \subseteq Jf$, then $C(X)$ denotes the set of path components of X ordered from right to left. And Jf is always assumed to have the natural order of the reals. A point of $\alpha_f J$ contained in $\hat{J}f$ will be referred to by its Y-coordinate.

Lemma 1.6. *Let* $h\colon \alpha_f J \to \alpha_g J$ *be a homeomorphism. Then the following hold:*

(i) $h|\hat{J}f$ *is a monotone homeomorphism onto* $\hat{J}g$.

(ii) *If* $\{[p_i, q_i]_f\}$ *is a sequence of wedges converging to* $[s, s+\varepsilon] \subseteq \hat{J}f$ *with* $\lim\{p_i\} = \lim\{q_i\} = s + \varepsilon$, *and if*

$h|\hat{J}f$ *is order preserving (reversing), then* $\{[h(p_i),h(q_i)]_g\}$ *is almost a sequence of wedges (spikes).*

(iii) *If* $\{[p_i,q_i]_f\}$ *is a sequence of spikes converging* $[s,s-\varepsilon] \subseteq \hat{J}f$, *with* $\lim\{p_i\} = \lim\{q_i\} = s - \varepsilon$, *and if* $h|\hat{J}f$ *is order preserving (reversing), then* $\{[h(p_i),h(q_i)]_g\}$ *is almost a sequence of spikes (wedges).*

Proof.

(i) is immediate.

(ii) Since h is a homeomorphism, $h[p_i,q_i]_f = [h(p_i),h(q_i)]_g$ and the sequence $\{[h(p_i),h(q_i)]_g\}$ converges to $[h(s),h(s+\varepsilon)]$. For each i, let m_i be a lowest point in $[h(p_i),h(q_i)]_g$, then

$$\lim\{h(p_i)\} = \lim\{h(q_i)\} = h(s+\varepsilon) > h(s) = \lim\{m_i\}.$$

Hence almost each m_i is an interior point of $[h(p_i),h(q_i)]_g$. Since m_i was an arbitrary lowest point of $[h(p_i),h(q_i)]_g$, this implies that in almost each $[h(p_i),h(q_i)]_g$ the lowest points are interior points. Hence the desired result follows. The case when $h|\hat{J}f$ is order reversing is handled similarly. The proof of (iii) is similar to that of (ii).

2. A Reduction Theorem

Definition 2.1. Let V be a decreasing sequence of points in J converging to 0, and let f ∈ F. Then f is called *piecewise linear* over $Vf = \{(v,f(v)): v \in V\}$ if $f|[v,v']$ is linear for each pair, v,v', of consecutive elements of V. If no ambiguity arises, f is called simply P.L. The set Vf is called the *set of vertices of* f. Moreover if f is P.L. and every P ∈ Vf is a local extremum, then f is called *sawtooth*. Finally let mf and Mf denote, respectively, the local minima and local maxima of f.

Remark 2.2. Let $f \in F$ be P.L. over Vf, then $V = \pi_x(Vf)$ is a copy of the integers (π_x is the projection on the x-coordinate). Hence $f|V\colon V \to I$ also yields a compactification of the integers, $\alpha_f V = \overline{Vf}$. $\hat{V}f$ denotes the remainder $\alpha_f V \backslash Vf$.

In this section we prove the following:

Reduction Theorem 2.3. *For each* $f \in F$, *there exists* $g \in F$ *having the following properties:*

(i) g *is sawtooth*

(ii) $\alpha_f J \cong \alpha_g J$ *and* $\hat{J}f = \hat{J}g$

(iii) $Ef = Eg = \hat{V}g$

A particularly nice property of the above sawtooth function g is that $Eg = \hat{V}g$. Although it can be predicted from 1.3 and the proof of 1.6 that $Eg \subseteq \hat{V}g$, equality is not in general true. In fact, Eg may consist of just two points, whereas $\hat{V}g$ may be all of I. So the above theorem implies some sort of minimality about the vertices Vg, in the sense that if a subsequence of Vg converges to a point s, the $s \in Eg$.

Lemma 2.4. *Let* $f,g \in F$ *such that* $\lim_{x\to 0}|f(x) - g(x)| = 0$. *Then the function* $h\colon \alpha_f J \to \alpha_g J$ *given by*

(i) $h|\hat{J}f = id$

(ii) $h(x,f(x)) = (x,g(x))$

is a homeomorphism.

Lemma 2.5. *Let* $f \in F$ *be* P.L. *Then there exists a sawtooth function* $g \in F$, *such that* $\alpha_f J \cong \alpha_g J$.

Proof. Let Vf be the set of vertices of f, and let $\pi_x(Vf) = V$. Choose $a_1 = (1, f(1))$, and let $a_1 > a_2 > a_3 > \cdots$ be a sequence of points in V, such that for each i, f is monotone over $[a_i, a_{i+1}]$ and is not monotone over any subinterval of J properly containing $[a_i, a_{i+1}]$. Clearly $\cup_{i=1}^{\infty}[a_i, a_{i+1}] = J$. Let g be the P.L. function over $\{(a_i, f(a_i))\}$. Then g is sawtooth, since each a_i is a local extremum of g. In order to prove that $\alpha_f J \cong \alpha_g J$, we construct another function $g_1 \in F$ as follows:

(i) $g_1(a_i) = f(a_i) = g(a_i)$ for each i.

(ii) $g_1|(a_i, a_{i+1})$ is a strictly monotone function such that $|f(x) - g_1(x)| < \varepsilon_i$, where the sequence $\{\varepsilon_i\}$ is a decreasing sequence of positive numbers converging to 0. This is possible because $f|(a_i, a_{i+1})$ is monotone for each i. Since $\lim\{\varepsilon_i\} = 0$, it follows that $\lim_{x \to 0}|f(x) - g_1(x)| = 0$. By Lemma 2.4, $\alpha_f J \cong \alpha_{g_1} J$. Finally define a function $h: \alpha_{g_1} J \to \alpha_g J$ as follows:

(i) $h|Jg_1 = \text{id}$, since $\hat{J}g_1 = \hat{J}g$.

(ii) Let $P_i = (a_i, g_1(a_i)) = (a_i, g(a_i))$ for each i. Then h maps $[P_i, P_{i+1}]_{g_1}$ onto $[P_i, P_{i+1}]_g$ by the horizontal projection: $h(x, g_1(x)) = (x', g(x'))$ where $g_1(x) = g(x')$. h is 1-1 on each $[P_i, P_{i+1}]_{g_1}$ because both g_1 and g are strictly monotone over $[a_i, a_{i+1}]$. One readily verifies that h is a homeomorphism.

Proof of Theorem 2.3. We break the proof into steps:

Step 1. Let $g_1 \in F$ be a P.L. function such that $\alpha_f J$ is homeomorphic to $\alpha_{g_1} J$ and where $\pi_X(Vg_1) \quad _1)$ converges to 0. By Lemma 2.5, we may assume that g_1 is sawtooth. It follows from Definition 1.3 that Eg_1 is closed in $\hat{J}g_1$. Hence $\hat{J}g_1 \backslash Eg_1$ is the countable (possibly finite) union of disjoint open intervals (t_i, s_i), $t_i < s_i$. For each i, choose positive numbers ℓ_i, k_i, r_i, so that $s_i > \ell_i > k_i > r_i > t_i$, and let $U_i = ((t_i, s_i) \times I) \cap Jg_1$.

Step 2. A new function $g_2 \in F$ is obtained from g_1 by altering Jg_1 over each U_i separately. Any portion of Jg_1 which is not altered is assumed to stay as part of Jg_2. We alter a typical U_i by considering each $K \in C(U_i)$ separately. If $K \in C(U_i)$ lies totally in one of the strips $k_i \le y \le s_i$; $t_i \le y \le k_i$, then K is left intact. Otherwise, removing the line $y = k_i$ splits K into at least two components. The closure of a typical such component $[p_0, q_0]_{g_1}$ is one of the following types:

(i) a wedge (spike) contained in the strip $k_i \ge y \ge r_i (k_i \le y \le \ell_i)$. In this case, replace $[p_0, q_0]_{g_1}$ by $[p_0, q_0]$.

(ii) $[p_0, q_0]_{g_1}$ is neither a wedge nor a spike, and is either totally contained in the strip $k_i \le y \le s_i$ or the strip $t_i \le y \le k_i$. We treat the two cases simultaneously, the alternative choices for the second case being included in parenthesis. We may assume that $\pi_2(p_0) > \pi_2(q_0)$. Before we alter $[p_0, q_0]_{g_1}$, we obtain a sequence $\{p_1, \cdots, p_j\}(\{q_1, \cdots, q_j\})$ of vertices in $[p_0, q_0]_{g_1}$ such that for each k, $1 \le k \le j$, p_k (q_k) is a highest (lowest) vertex in the interior of $[p_{k-1},q_0]_{g_1}$ ($[q_{k-1},p_0]_{g_1}$) and such that $[p_j,q_0]_{g_1}$ ($[q_j,p_0]_{g_1}$) contains no vertices of g_1. Also, for each k, $1 \le k \le j$, $[p_k,Q_k]_{g_1}$ ($[q_k,Q_k]_{g_1}$) is chosen to be the largest symmetric wedge (spike) in $[p_k,p_0]_{g_1}$ ($[q_k,q_0]_{g_1}$) having p_k (q_k) as one endpoint. Now $[p_0,q_o]_{g_1}$ is altered by replacing each $[p_k,Q_k]_{g_1}$ ($[q_k,Q_k]_{g_1}$) by $[p_k,Q_k]$ ($[q_k,Q_k]$).

(iii) Finally if $[p_0,q_0]_{g_1}$ is a wedge (respectively, spike) that extends below the line $y = r_i$ (above the line $y = \ell_i$), then choose a lowest (highest) vertex M in $[p_0,q_0]_{g_1}$. The arcs $[p_0,M]_{g_1}$, $[M,q_0]_{g_1}$ are both the types discussed in (ii) and are altered accordingly.

Step 3. $\lim_{x\to 0}|g_1(x) - g_2(x) = 0$. To establish this, let $\{[p_i,q_i]_{g_1}\}$ be an enumeration from right to left of all the wedges in Jg_1 that have been altered in a particular U_j. And let $\{[p_i',q_i']_{g_1}\}$ be the similar sequence of all the spikes. Our claim is equivalent to showing that each of $\{[p_i,q_i]_{g_1}\}$ and $\{[p_i',q_i']_{g_1}\}$ converges to a point. Suppose $\{[p_i,q_i]_{g_1}\}$ does not converge to a point, then one deduces that a point of $[k_j,s_j)$ is essential. Similarly if $\{[p_i',q_i']_{g_1}\}$ does not converge to a point, then a point of $(t_j,k_j]$ is essential. Both of these conclusions contradict the assumption that (t_j,s_j) consists of inessential points.

Step 4. Applying Lemma 2.5 to g_2 we obtain the desired sawtooth function g. To see this let $h_1: \alpha_f J \to \alpha_{g_1} J$, $h_3: \alpha_{g_2} J \to \alpha_g J$ be the homeomorphisms guaranteed by Lemma 2.5, and let $h_2: \alpha_{g_1} J \to \alpha_{g_2} J$ be the homeomorphism guaranteed by Lemma 2.4. Then the composition $h_3 \cdot h_2 \cdot h_1: \alpha_f J \to \alpha_g J$ is a homeomorphism. Also $h|\hat{J}f = \mathrm{id}$. This and Theorem 1.3 establish (i), (ii) and half of (iii) of Theorem 2.3.

Finally to prove that $\hat{V}g = Eg$, we notice that the sequence $Vg \cap ([k_j, s_j) \times I)$ converges to s_j and the sequence $Vg \cap ((t_j, k_j] \times I)$ converges to t_j, otherwise a point of $[k_j, s_j)$ or $(t_j, k_j]$ would be essential.

3. Homeomorphism Classes of Compactifications of a Given Type

The type Ef of a compactification $\alpha_f J$ is not a complete invariant. In this section, we prove the following:

Theorem 3.1. *Let* T *be a closed subset of* I *containing more than two points. Then there exists continuum many nonhomeomorphic compactifications of the ray all of which are of type* T.

In the case where the type Ef consists of two points, using a procedure similar to that of Theorem 2.3, one obtains a reduction that "straightens" Jf enough so that the resulting graph looks like that in figure 2. Hence all compactifications of J whose type consists of two points are homeomorphic to $\alpha_{\sin\frac{1}{x}} J$.

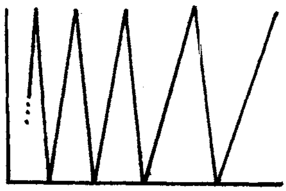

Figure 2

The strategy for proving the above theorem involves associating (in 3.3) with each nonempty set A of positive integers, a function $f \in F$ where $\alpha_f J$ is of type T, and then proving that functions associated with different sets of positive integers yield nonhomeomorphic compactifications, all of which are of type T.

Before we proceed with the proof of Theorem 3.1, we need the following:

Definition 3.2. Let S be a totally ordered sequence. A *block* in S is a finite set of consecutive elements of S. If the cardinality of b, $|b| = n$, then b is an *n-block* in S. *The boundary of* b *in* S, $bd_S(b)$, is a subset of $S\backslash b$ consisting of two elements: the element preceding the first element of b, and that succeeding the last element of b. A subsequence S_1 of S is called an *n-subsequence* of S, for a positive integer n, if $S_1 = \cup_{i=1}^{\infty} b_i$, such that almost each b_i is an n-block in S, with $bd_S(b_i) \subset S\backslash S_1$. *The boundary of* S_1 *in* S, $bd_S(S_1) = \cup_{i=1}^{\infty} bd_S(b_i)$.

Observe that if S_1 is an n-subsequence of S and $m \neq n$, then S_1 is not an m-subsequence of S.

Construction 3.3. Let T be the set specified in Theorem 3.1 and let t,t' be respectively the smallest and largest elements of T, and let D be a countable dense subset of T\{t'}, containing t. Given a nonempty set A of positive integers we associate with A a sawtooth function f ∈ F whose vertices Vf satisfy the following conditions:

(i) All elements of Mf are at height t'.

(ii) For each d ∈ D, and each a ∈ A, the minima of f at height d, denoted by md, contain an a-subsequence S of mf with $bd_{mf}(S) \subseteq mt$.

(iii) Moreover, if b is a block in mf contained in md, with $bd_{mf}(b) \subseteq mt$, then |b| ∈ A.

To illustrate the above construction, let $T = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$: A = {1,2}. The function in figure 3 satisfies the conditions of Construction 3.3

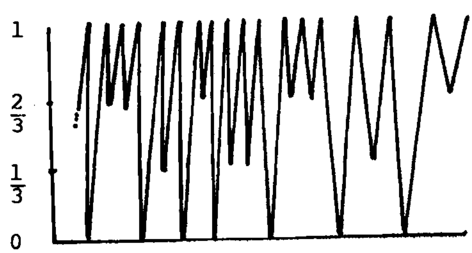

Figure 3

Lemma 3.4. *Let* f *be the function constructed in* 3.3, *and let* S *be an* n-*subsequence of* mf *which converges to* d ∈ D\{t}, *and whose boundary,* $bd_{mf}(S)$, *converges to* t. *Then* n ∈ A.

Proof. Since S is an n-subsequence of mf, $S = \cup_{i=1}^{\infty} b_i$, where almost each b_i is an n-block in m_f and $bd_{mf}(b_i) \subseteq mf\backslash S$.

Since lim S $\neq$ t, almost each $b_i \subseteq mf\backslash mt$. It follows then from part (ii) of construction 3.3, that an element of $bd_{mf}(b_i)$ is either at the same height as elements of b_i, or is contained in mt. Since $bd_{mf}(S)$ converges to t $\neq$ d, it follows that almost each $bd_{mf}(b_i)$ is contained in mt. Hence part (iii) of Construction 3.3 implies that n $\in$ A.

Proof of Theorem 3.1. Let A and B be two sets of positive integers, and let f,g be the function associated with A,B respectively. Observe that $\hat{J}f = \hat{J}g = [t,t']$. Suppose that A $\neq$ B, and suppose that h: $\alpha_f J \to \alpha_g J$ is a homeomorphism. We may assume that there exists an a $\in$ A\B. Choose d $\in$ D\{t}, and let U = ([t,t') × I) $\cap$ Jf. Then by part (ii) of Construction 3.3, there exists a sequence S in C(U) having the following properties:

(i) S is an a-subsequence of C(U) converging to [d,t'].

(ii) $bd_{C(U)}(S)$ converges to [t,t'].

(iii) each K $\in$ C(U) contains exactly one element of mf.

Since h is a homeomorphism, it is an order isomorphism taking C(U) onto C(h(U)). Now we verify the following:

a) $h|\hat{J}f$ is order preserving. Suppose the contrary and consider the above sequence S $\subseteq$ C(U). Since $h|\hat{J}f$ is order reversing, part (iii) of Lemma 1.6 implies that almost each L $\in$ h(S) is a spike. Consequently almost each L contains at least one element of Mg. Since h is a homeomorphism and since h(t') = t,h(S) must converge to [t,h(d)]. But this contradicts the fact that t' $\in$ lim h(S) since almost each L $\in$ h(S) contains at least one vertex of Mg, and Mg converges to t'.

b) It follows from (a) that h(S) is an a-subsequence of C(h(U)) converging to [h(d),t'] and $bd_{C(h(U))}(h(S))$ converges to [t,t'].

c) Almost each element L of h(S) or $bd_{C(h(U))}(h(S))$ contains exactly one element of mg. First, it follows from part (ii) of Lemma 1.6 and (a) above that almost each L is a wedge, and hence contains at least one element of mg. Suppose that S_1 is a subsequence of h(S) such that each element of S_1 contains at least two elements of mg. Then each such element L must contain at least one element M of Mg, which when deleted from L splits L into two arcs whose closures are both wedges. Whereas if $h^{-1}(M)$ is deleted from $h^{-1}(L) \in S$, then clearly $h^{-1}(L)$ breaks up into two arcs such that the closure of at most one of them is a wedge. This contradicts part (ii) of Lemma 1.6. Similarly, we prove that almost each $L \in bd_{C(h(U))}(h(S))$ contains exactly one element of mg.

d) It follows from (b) and (c) above that the sequence $h(S) \cap mg$ is an a-subsequence of mg which converges to h(d) and whose boundary in mg converges to t. Hence by Lemma 3.4, $a \in B$, contradicting the fact that a was chosen in A\B, and hence proving that $\alpha_f J$ and $\alpha_g J$ are nonhomeomorphic.

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