A METRIC FOR METRIZABLE
GO-SPACES

by

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Conditions which force the metrizability of GO-spaces are well known (see [Fa]). Since GO-spaces are $T_3$-spaces and countable GO-spaces are first countable it follows that countable GO-spaces are metrizable. However it is not always apparent what a metric is for a given metrizable GO-space even if it is countable. For example the Sorgenfrey line [S] restricted to the set of rational numbers or, if $a < \omega_1$, the LOTS $[0,a]$ are both countable and, thus, metrizable but it is difficult to construct a metric for either of these spaces ([A]). In this note a metric is derived for GO-spaces.

A LOTS (= linearly ordered topological space) is a triple $(X, \lambda(\leq), \leq)$ where $(X, \leq)$ is a linearly ordered set and $\lambda(\leq)$ is the usual open-interval topology generated by the order $\leq$.

Recall that a subset $A$ of $X$ is order-convex if whenever $a$ and $b$ are in $A$, then each point lying between $a$ and $b$ is also in $A$.

A GO-space (= generalized ordered space) is a subspace of a LOTS (see [L]). There is an equivalent way to obtain a GO-space $X$ by starting with a linearly-ordered set $Y$. Equip $Y$ with a topology $\tau$ that contains $\lambda(\leq)$ and has a base of open sets each of which is order-convex. In this case $X$ is said to be constructed on $Y$ and $X = \text{GO}_Y(R,E,I,L)$ where
In deriving a metric for a metrizable GO-space it is illuminating to first derive a metric for a countable GO-space case and then derive the metric for an arbitrary metrizable GO-space.

Let \( Q \) denote the LOTS of rational numbers and let \( N \) denote the set of natural numbers. An order \( \leq \) on a set \( X \) is a dense-order if whenever \( a, b \in X \) are such that \( a < b \) then there is a point \( c \) in \( X \) such that \( a < c < b \).

The next theorem indicates how a GO-space may be embedded in a LOTS.

**Theorem 1.** If \( X = GO(R, E, I, L) \), then \( X \) is homeomorphic to a subspace of a dense-ordered LOTS \( L(X) \). Furthermore the homeomorphism is order-preserving and \( L(X) \) does not have any endpoints.

**Proof.** Let

\[
L(X) = \{(x, q) | x \in I, q \in ]-1,1[ \cap Q \} \cup \\
\{(x, q) | x \in R, q \in ]-1,0[ \cap Q \} \cup \\
\{(x, q) | x \in L, q \in [0,1[ \cap Q \} \cup \\
\{(x, 0) | x \in E \}.
\]

Equip \( L(X) \) with the lexicographic ordering induced from the order on \( X \) and the natural order on \( Q \). It follows that \( L(X) \) is a dense-ordered LOTS without endpoints. Define a function \( \phi \) from \( X \) into \( L(X) \) by \( \phi(x) = (x, 0) \). Then \( \phi \) is an order-preserving homeomorphism from \( X \) into \( L(X) \).
Corollary. A countable GO-space $X$ is homeomorphic to a subspace of $Q$ by an order-preserving homeomorphism.

Proof. Since $X$ is countable, $L(X)$ is a countable, dense-ordered LOTS without endpoints. Hence $L(X)$ is homeomorphic to $Q$ by an order-preserving homeomorphism [Fr]. Thus $X$ is homeomorphic to a subspace of $Q$ by an order-preserving homeomorphism.

Since each countable GO-space $X$ can be considered a subspace of $Q$ the usual metric on $Q$ restricted to $X$ is a metric on $X$. Unfortunately it is often difficult to use this metric since it is hard to visualize the embedding.

Let $X$ be a countable GO-space and $\phi$ be an order-preserving homeomorphism from $X$ into $L(X)$ and $\beta$ be an order-preserving homeomorphism from $L(X)$ onto $Q$. Notice that if $x \in R$ ($x \in L$) then there is an interval $J$ in $L(X)$ immediately preceding (succeeding) $\phi(x)$ such that no point of $X$ maps into $\beta(J)$ and if $x \in I$ then there are intervals $J_1$ and $J_2$ in $L(X)$ such that $J_1$ immediately precedes $\phi(x)$ and $J_2$ immediately succeeds $\phi(x)$ and no point of $X$ maps into $\beta(J_1 \cup J_2)$. By considering $Q$ homeomorphic to $Q \cap [0,1[$ and embedding $X$ in $Q \cap [0,1[$ it follows that the image of those intervals in $Q \cap [0,1[$ must be made arbitrarily small if $|R \cup L \cup I|$ is large.

Let $K$ be the collection of all maximal, nondegenerate, convex subsets of $X - (R \cup L \cup I)$. Then $K$ is at most countable. Let $K_1, K_2, \ldots$ be an enumeration of $K$ (without repetitions). Since each $K_i$ is homeomorphic to a convex subset of $Q$ it is metrizable. Let $d_i$ be a metric for $K_i$.
that is bounded by 1. Let \( x_1, x_2, \ldots \) be a counting of \( R \cup L \cup I \).

These observations motivate the derivation of a metric for a countable GO-space.

To define the function on \( X \times X \), compensation functions must be defined for the points of \( X \). The motivation for these compensation functions comes from observing how \( X \) is embedded in \( Q \) and how one would "travel" in \( Q \) from point to point. Let

\[
\phi_\ell(x) = \begin{cases} 
2^{-n} & \text{if } x = x_n \in R \cup I \\
0 & \text{if } x \in L \cup E 
\end{cases}
\]

and

\[
\phi_r(x) = \begin{cases} 
2^{-n} & \text{if } x = x_n \in L \cup I \\
0 & \text{if } x \in R \cup E. 
\end{cases}
\]

A metric function \( \sigma \) can be defined on \( X \times X \). Although it is not necessary it is convenient to consider cases. Let \( a < b \).

Case 1. If \( \{a, b\} \subseteq UI \) and both lie in the same \( K_i \) let

\[
\sigma(a, b) = d_i(a, b) \cdot 2^{-1}
\]

and if \( a \in K_i \) and \( b \in K_j \) for \( i \neq j \), then let

\[
\sigma(a, b) = \sup \{2^{-1} \cdot d_i(a, z) \mid a < z, z \in K_i\} + \\
\varepsilon\{2^{-n} \mid K_n \subseteq [a, b]\} + \\
\varepsilon\{\phi_\ell(x) + \phi_r(x) \mid a < x < b\} + \\
\sup \{2^{-j} \cdot d_j(z, b) \mid z < b, z \in K_j\}.
\]

Case 2. If \( a \in K_j \) and \( b \notin UI \) then let

\[
\sigma(a, b) = \sup \{2^{-j} \cdot d_j(a, z) \mid a < z, z \in K_j\} + \\
\varepsilon\{2^{-n} \mid K_n \subseteq [a, b]\} + \\
\varepsilon\{\phi_\ell(x) + \phi_r(x) \mid a < x < b\} + \\
\phi_\ell(b).
\]
If \( a \not\in \cup K \) and \( b \in K_j \), then let
\[
\sigma(a,b) = \phi_\bar{x}(a) + \sum \{2^{-n}|K_n \subseteq \{a,b\} + \\
\sum \{\phi_k(x) + \phi_\bar{x}(x)|a < x < b\} + \\
\sup \{2^{1-n} \cdot d_\bar{z}(z,b)|z < b, z \in K_i\}.
\]

**Case 3.** If neither \( a \) nor \( b \) is in \( \cup K \), then let
\[
\sigma(a,b) = \phi_\bar{x}(a) + \sum \{2^{-n}|K_n \subseteq \{a,b\} + \\
\sum \{\phi_k(x) + \phi_\bar{x}(x)|a < x < b\} + \\
\phi_k(b).
\]
Furthermore let \( \sigma(a,b) = 0 \) if and only if \( a = b \) and let \( \sigma(a,b) = \sigma(b,a) \) for \( a \) and \( b \) in \( X \).

**Theorem 1.** If \( X \) is a countable GO-space, then \( \sigma \) is a metric on \( X \).

**Proof.** Since each of the series used in defining \( \sigma \) is bounded by the convergent series \( 2 \cdot \sum 2^{-n} \), it follows that \( \sigma \) is well-defined. Since \( \sigma \) was constructed to be a metric function it is just a matter of cases to check that \( \sigma \) defines the topology. Let \( S_\sigma(x,\varepsilon) \) denote the sphere centered at \( x \) whose \( \sigma \)-radius is \( \varepsilon \).

**Case 1.** If \( x_n \in I \), then \( S_\sigma(x_n,2^{-n}) = \{x_n\} \).

**Case 2.** Let \( x_n \in R \) and \( S_\sigma(x_n,\varepsilon) \) be given. Since \( x_n \in R \) choose \( x \in X \) such that \( x_n < x \) and \( x \in S_\sigma(x_n,\varepsilon) \). Then \( [x_n,x[ \subseteq S_\sigma(x_n,\varepsilon) \).

Let \( [x_n',x[ \) be given. If \( \sigma(x_n',x) = \varepsilon_\bot \), let
\[
\varepsilon = \min\{2^{-n},\varepsilon_\bot\}.
\]
Then \( [x_n',x[ \supset S_\sigma(x_n',\varepsilon) \).

**Case 3.** If \( x_n \in L \) argue analogously to Case 2.
Case 4. If $x_n \in E$ then argue on each side of $x_n$ using Case 2 and Case 3.

Hence $\sigma$ is a metric for $X$.

If $R \cup L \cup I$ is dense in $X$ then the metric $\sigma$ is much less simple as the next two examples illustrate.

Example 1. Let $X$ be the Sorgenfrey Line restricted to $Q$, that is, $GO_Q(Q, \phi, \phi, \phi)$. Let $q_1, q_2, \cdots$ be a counting of $Q$. Then, for each $k \in \mathbb{N}$, $\phi_q(q_k) = 2^{-k}$ and $\phi_r(q_k) = 0$.

Thus if $q_n < q_m$, it follows that

$$\sigma(q_n, q_m) = \Sigma(2^{-k} | q_n < q_k < q_m).$$

Example 2. Let $X$ be the LOTS $[l, a)$ where $a < \omega_1$. Let $x_1, x_2, \cdots$ be a counting of $[l, a)$. Then if $x_\xi$ is a non-limit ordinal $\phi_r(x_\xi) = \phi_\xi(x_\xi) = 2^{-\xi}$ and if $x_\xi$ is a limit ordinal $\phi_\xi(x_\xi) = 0$ and $\phi_r(x_\xi) = 2^{-n}$. Hence, if $x_n < x_m$, then

$$\sigma(x_n, x_m) = \phi_r(x_n) + \Sigma(\phi_r(x) + \phi_\xi(x) | x_n < x < x_m) + \phi_\xi(x_m).$$

The following corollary easily follows.

Corollary. If $R \cup L \cup I$ is dense in the countable GO-space $X$ then

$$\sigma(a, b) = \phi_r(a) + \Sigma(\phi_r(x) + \phi_\xi(x) | a < x < b) + \phi_\xi(b)$$

is a metric for $X$.

The countable GO-space case motivates the metrizable GO-space case by realizing the countable GO-spaces are $\sigma$-discrete.
The following theorem gives structural conditions for the metrizability of a given GO-space.

**Theorem 2 [Fa].** Let $X$ be a GO-space. The following properties are equivalent.

(i) $X$ is metrizable, and

(ii) There is a dense, $\sigma$-discrete set $D$ in $X$ containing $R \cup L$.

It follows from this result that if $X$ is a metrizable GO-space then each of $R$ and $L$ are $\sigma$-discrete in $X$. Since $I$ is open in $X$ it is an $F_\sigma$-set and, hence, a $\sigma$-discrete set.

Let $R = \bigcup \{R_n | n = 1, 2, \ldots \}$, $L = \bigcup \{L_n | n = 1, 2, \ldots \}$ and $I = \bigcup \{I_n | n = 1, 2, \ldots \}$ where for each $n$, $R_n \subseteq R_{n+1}$, $L_n \subseteq L_{n+1}$ and $I_n \subseteq I_{n+1}$.

If $X = GO_y(R,E,I,L)$ is a metrizable GO-space where $Y$ is a metric LOTS with metric $d$ then in order to find a metric for $X$ compensation functions must be found (as in the countable case). This is motivated by embedding $X$ in $L(X)$ and observing how one "travels" from point to point. If $x \leq y$, let $R(x,y) = 2^{-i}$, where $i$ is the first natural number such that $R_i \cap [x,y] \neq \emptyset$. If no such $i$ exists let $R(x,y) = 0$. Let $L(x,y) = 2^{-j}$ where $j$ is the first natural number such that $L_j \cap [x,y] \neq \emptyset$. If no such $j$ exists let $L(x,y) = 0$. If $x < y$ let $I(x,y) = 2^{-k}$ where $k$ is the first natural number such that $I_k \cap [x,y] \neq \emptyset$. If no such $k$ exists or if $x = y$ let $I(x,y) = 0$. 
Let
\[ \rho(y,x) = \rho(x,y) = d(x,y) + R(x,y) + L(x,y) + I(x,y). \]
It is a matter of checking cases to see that \( \rho \) is a metric function on \( X \). Notice if \( y_1 < y_2 \) and \( x < y_1 \) then
\[ \rho(x,y_1) \leq \rho(x,y_2). \]

**Theorem 2.** Let \( Y \) be a LOTS with metric \( d \) and \( X = \text{GO}_Y(R,E,I,L) \) be a metrizable \( G \)-space. Then \( \rho \), defined above, is a metric on \( X \).

**Proof.** All that needs to be shown is that \( \rho \) preserves the topology on \( X \). Consider the following cases:

(i) If \( x \in I \) then let \( k \) be the first natural number such that \( x \notin I_k \). It follows that \( S_\rho(x,2^{-k}) = \{x\} \).

(ii) If \( x \in R \) and \( S_\rho(x,\epsilon) \) is given, choose the first natural number \( n \) such that \( 3 \cdot 2^{-n} < \epsilon \cdot 2^{-2} \). Let \( K_n = \bigcup \{ R_i \cup L_i \cup I_i \mid i = 1, \ldots, n \} \). Choose \( y > x \) such that \( d(x,y) < \epsilon \cdot 2^{-2} \) and \( \|x,y\| \cap K_n = \emptyset \). This can be done since \( K_n \) is discrete and \( x \in R \). It follows that \( \rho(x,y) < \epsilon \cdot 2^{-2} + 3 \cdot \epsilon \cdot 2^{-2} = \epsilon \). Thus \( \|x,y\| \in S_\rho(x,\epsilon) \).

If \( \{x,b\} \) is given let \( n \) be the first natural number such that \( x \in R_n \). Let \( \epsilon = \min\{d(x,b),2^{-n}\} \). Then \( S_\rho(x,\epsilon) \subseteq \{x,b\} \).

(iii) If \( x \in L \) argue analogously to (ii).

(iv) If \( x \in E \) combine (ii) and (iii).

Hence \( \rho \) preserves the topology on \( X \) and, hence, is a metric for \( X \).
Corollary. If \( R \cup L \cup I \) is dense in the metrizable GO-space \( X \) then

\[
\rho(x, y) = R(x, y) + L(x, y) + I(x, y)
\]

is a metric on \( X \).

Let \( E \) denote the real line with the usual order topology.

Example 3. Let \( X = \text{GO}_E(Q, E - Q, \phi, \psi) \) and let \( q_1, q_2, \ldots \) be any counting of the rational numbers. Then

\[
\rho(x, y) = R(x, y) = 2^{-j}
\]

(where \( q_j \) is the first rational number in \( ]x, y[ \)) is a metric on \( X \).

If \( Y = \text{GO}_Q(Q, \phi, \psi) \) (i.e., the Sorgenfrey Line) then the above \( \rho \) is a metric on \( Y \) that is simpler than the metric given in Example 1.

References


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