A FACTORIZATION OF THE
COLLINS-ROSCOE THEOREM

by

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In [CR] Collins and Roscoe proved the following metrization theorem:

Theorem 1. [CR] In order that a $T_1$-space be metrizable it is necessary and sufficient that, for each $x \in X$, there is a countable decreasing local neighborhood basis $\{W(i,x)\mid i \in \mathbb{N}\}$ (where $\mathbb{N}$ denotes the set of natural numbers) satisfying

(i) if $x \in U$ and $U$ is an open set, then there exists a natural number $n = n(x,U)$ and an open set $V = V(x,U)$ containing $x$ such that $x \in W(n,y) \subseteq U$ whenever $y \in V$.

In this note the Collins-Roscoe theorem is factored so one can topologically see why the result holds.

Definition 1. A countable local neighborhood basis for a space $X$ is a collection $\mathcal{W} = \{W(i,x)\mid i \in \mathbb{N}, x \in X\}$ of not necessarily open sets such that

(i) For each $i \in \mathbb{N}$ and $x \in X$, $x \in W(i,x)_0$, and

(ii) If $x$ is in an open set $U$, then there exists $n = n(x,U) \in \mathbb{N}$ such that $W(n,x) \subseteq U$.

Definition 2. A space $X$ is quasi-developable [Be] if there is a sequence $\mathcal{G} = \{G_1, G_2, \cdots\}$ of collections of open sets such that if $x$ is in an open set $U \subseteq X$, then there exists $n(x,U) = n \in \mathbb{N}$ such that $\text{st}(x,G_n) \subseteq U$. The sequence $\mathcal{G}$ is a quasi-development for $X$. 
All undefined terms and concepts are as in [E]. All spaces are $T_1$-spaces.

Consider the following conditions on a countable local neighborhood bases $W = \{ W(i,x) | i \in \mathbb{N}, x \in X \}$ for a space $X$:

A(1). Given $W(i,x)^O$ there is an open set $V(i,x)$ containing $x$ and a natural number $b(i,x) > i$ such that if $y \in V(i,x)$, then $x \in W(b(i,x),y)$,

A(2). Given $W(i,x)^O$ there is an open set $V(i,x)$ containing $x$ and a natural number $b(i,x)$ such that if $y \in V(i,x)$, then $x \in W(b(i,x),y) \subseteq W(i,x)^O$,

A(3). For each $x \in X$ and $i \in \mathbb{N}$, $W(i+1,x) \subseteq W(i,x)$.

The Michael Line $[M]$ satisfies A(1) and A(2) but not A(3). Heath's plane $[H]$ satisfies A(1) and A(3) but not A(2).

It is clear that if the hypothesis of the Collins-Roscoe Theorem is assumed on a countable local neighborhood base then A(2) and A(3) are satisfied. To see that A(1) is satisfied let $W(i,x)^O$ be given. Choose $j(i)$ to be the first natural number such that $W(j(i),x)^O$ is properly contained in $W(i,x)^O$. Let $V(i,x) = V(x,W(j(i),x)^O)$ and $b(i,x) = n(x,W(j(i),x)^O)$. If $x \in V(i,x)$, then it follows that $x \in W(b(i,x),x)^O \subseteq W(b(i,x),x) \subseteq W(j(i),x)^O$. Thus $W(b(i,x),x) \subseteq W(i,x)^O \subseteq W(i,x)$. Since the local neighborhood base is decreasing it must follow that $b(i,x) > i$.

**Theorem 2.** Let $X$ be a $T_1$-space with a countable local neighborhood basis $W$. Then

(i) If $W$ satisfies A(1) and A(3), then closed subsets of $X$ are $G_\delta$-sets,
(ii) If \( \mathcal{W} \) satisfies \( A(2) \), then \( X \) is a quasi-developable space, and

(iii) If \( \mathcal{W} \) satisfies \( A(2) \) and \( A(3) \) \( X \) is a collection-wise normal space.

Proof. (i) This follows immediately from Theorem 11 of [CR].

(ii) Let \( \mathcal{W} \) be a local neighborhood basis for \( X \) that satisfies \( A(2) \). Let \( G_0 = \{\{x\}|\{x\} \text{ is open in } X\} \). Arbitrarily fix \( i \in \mathbb{N} \). For each \( x \in X \), \( W(i,x)^0 \) induces an open set \( V_1(i,x) \) containing \( x \) and a natural number \( b_1(i,x) \) such that if \( y \in V_1(i,x) \) then \( x \in W(b_1(i,x),y) \subseteq W(i,x)^0 \). Notice that \( V_1(i,x) \subseteq W(i,x)^0 \). Choose \( m(i,x) \in \mathbb{N} \) such that \( W(m(i,x),x) \subseteq V_1(i,x) \cap W(b_1(i,x),x)^0 \).

Then \( W(m(i,x),x)^0 \) induces an open set \( V_2(i,x) \) containing \( x \) and a natural number \( b_2(i,x) \) such that if \( y \in V_2(i,x) \) then \( x \in W(b_2(i,x),y) \subseteq W(m(i,x),x)^0 \).

Let \( G(i,j,k,\ell) = \{V_2(i,x)|b_1(i,x) = j, m(i,x) = k, b_2(i,x) = \ell\} \), and let \( \mathcal{G} = \{G_0\} \cup \{G(i,j,k,\ell)|i,j,k,\ell \in \mathbb{N}^4\} \).

It follows that \( \mathcal{G} \) is a quasi-development for \( X \) since if \( \{x\} \) is open in \( X \) then \( st(x,G_0) \subseteq U \) where \( U \) is any open set containing \( x \). If \( \{x\} \) is not open in \( X \) and \( x \in U \) where \( U \) is open in \( X \) then choose \( i \in \mathbb{N} \) such that \( W(i,x) \subseteq U \).

Let

\[ V_2(i,z) \in G(i,b_1(i,x), m(i,x), b_2(i,x)) \]

such that \( x \in V_2(i,z) \). Thus \( z \in W(b_2(i,z),x) \) and since \( b_2(i,z) = b_2(i,x) \), it follows that

\[ x \in W(b_2(i,x),x) \subseteq W(m(i,x),x)^0 \subseteq V_1(i,x) \].

Hence \( x \in W(b_1(i,x),z) \subseteq W(i,x)^0 \subseteq U \). Since
it follows that
\[ \text{st}(x, G(i, b_1(i, x), m(i, x), b_2(i, x)) \subseteq U \]
and \( X \) is a quasi-developable space.

(iii) This is noted in the remark following Theorem 11 of [CR] and specifically proved in Theorem 3 of [CRRR].

Clearly imposing conditions \( A(1), A(2) \) and \( A(3) \) and a local neighborhood bases \( W \) for a \( T_1 \)-space \( X \) is equivalent to the conditions in the Collins-Roscoe theorem. This represents the factorization of the Collins-Roscoe theorem.

**Proof of Theorem 1.** Since \( X \) has a countable local neighborhood basis satisfying \( A(1), A(2) \) and \( A(3) \) it is a quasi-developable space having closed sets \( G_\delta \)-sets and, thus, is developable [Be]. It is also a collection-wise normal \( T_1 \)-space and, hence, metrizable [Bi].

This factorization clearly shows the importance of the countable local neighborhood base being decreasing in the Collins-Roscoe Theorem. It also shows that a countable local neighborhood base that satisfies all the conditions of the Collins-Roscoe Theorem except being decreasing also implies a good deal of structure, i.e., quasi-developability, on the space.

**References**


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