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In [CR] Collins and Roscoe proved the following metrization theorem:

Theorem 1. [CR] *In order that a* T_1*-space be metrizable it is necessary and sufficient that, for each* $x \in X$*, there is a countable decreasing local neighborhood basis* $\{W(i,x) | i \in N\}$ *(where* N *denotes the set of natural numbers) satisfying*

(A) *if* $x \in U$ *and* U *is an open set, then there exists a natural number* $n = n(x,U)$ *and an open set* $V = V(x,U)$ *containing* x *such that* $x \in W(n,y) \subseteq U$ *whenever* $y \in V$.

In this note the Collins-Roscoe theorem is factored so one can topologically see why the result holds.

Definition 1. A countable local neighborhood basis for a space X is a collection $\mathcal{W} = \{W(i,x) | i \in N, x \in X\}$ of not necessarily open sets such that

(i) For each $i \in N$ and $x \in X$, $x \in W(i,x)^0$, and

(ii) If x is in an open set U, then there exists $n = n(x,U) \in N$ such that $W(n,x) \subseteq U$.

Definition 2. A space X is quasi-developable [Be] if there is a sequence $\mathcal{G} = \{G_1, G_2, \cdots\}$ of collections of open sets such that if x is in an open set $U \subseteq X$, then there exists $n(x,U) = n \in N$ such that $st(x,G_n) \subseteq U$. The sequence $\mathcal{G}$ is a quasi-development for X.

All undefined terms and concepts are as in [E]. All spaces are T_1-spaces.

Consider the following conditions on a countable local neighborhood bases $\mathcal{W} = \{W(i,x) \mid i \in N,\ x \in X\}$ for a space X:

A(1). Given $W(i,x)^{\circ}$ there is an open set $V(i,x)$ containing x and a natural number $b(i,x) \geq i$ such that if $y \in V(i,x)$, then $x \in W(b(i,x),y)$,

A(2). Given $W(i,x)^{\circ}$ there is an open set $V(i,x)$ containing x and a natural number $b(i,x)$ such that if $y \in V(i,x)$, then $x \in W(b(i,x),y) \subseteq W(i,x)^{\circ}$,

A(3). For each $x \in X$ and $i \in N$, $W(i+1,x) \subseteq W(i,x)$.

The Michael Line [M] satisfies A(1) and A(2) but not A(3). Heath's plane [H] satisfies A(1) and A(3) but not A(2).

It is clear that if the hypothesis of the Collins-Roscoe Theorem is assumed on a countable local neighborhood base then A(2) and A(3) are satisfied. To see that A(1) is satisfied let $W(i,x)^{\circ}$ be given. Choose $j(i)$ to be the first natural number such that $W(j(i),x)^{\circ}$ is properly contained in $W(i,x)^{\circ}$. Let $V(i,x) = V(x,W(j(i),x)^{\circ})$ and $b(i,x) = n(x,W(j(i),x)^{\circ})$. If $x \in V(i,x)$, then it follows that $x \in W(b(i,x),x)^{\circ} \subseteq W(b(i,x),x) \subseteq W(j(i),x)^{\circ}$. Thus $W(b(i,x),x) \subseteq W(i,x)^{\circ} \subseteq W(i,x)$. Since the local neighborhood base is decreasing it must follow that $b(i,x) \geq i$.

Theorem 2. *Let* X *be a* T_1*-space with a countable local neighborhood basis* $\mathcal{W}$. *Then*

(i) *If* $\mathcal{W}$ *satisfies* A(1) *and* A(3), *then closed subsets of* X *are* G_δ*-sets,*

(ii) *If* $\mathcal{W}$ *satisfies* A(2), *then* X *is a quasi-developable space, and*

(iii) *If* $\mathcal{W}$ *satisfies* A(2) *and* A(3) X *is a collection-wise normal space.*

Proof. (i) This follows immediately from Theorem 11 of [CR].

(ii) Let $\mathcal{W}$ be a local neighborhood basis for X that satisfies A(2). Let $G_o = \{\{x\} | \{x\}$ is open in $X\}$. Arbitrarily fix $i \in N$. For each $x \in X$, $W(i,x)^o$ induces an open set $V_1(i,x)$ containing x and a natural number $b_1(i,x)$ such that if $y \in V_1(i,x)$ then $x \in W(b_1(i,x),y) \subseteq W(i,x)^o$. Notice that $V_1(i,x) \subseteq W(i,x)^o$. Choose $m(i,x) \in N$ such that

$$W(m(i,x),x) \subseteq V_1(i,x) \cap W(b_1(i,x),x)^o.$$

Then $W(m(i,x),x)^o$ induces an open set $V_2(i,x)$ containing x and a natural number $b_2(i,x)$ such that if $y \in V_2(i,x)$ then

$$x \in W(b_2(i,x),y) \subseteq W(m(i,x),x)^o.$$

Let $G(i,j,k,\ell) = \{V_2(i,x) | b_1(i,x) = j,\ m(i,x) = k,\ b_2(i,x) = \ell\}$, and let $\mathcal{G} = \{G_o\} \cup \{G(i,j,k,\ell) | (i,j,k,\ell) \in N^4\}$.

It follows that $\mathcal{G}$ is a quasi-development for X since if $\{x\}$ is open in X then $st(x,G_o) \subseteq U$ where U is any open set containing x. If $\{x\}$ is not open in X and $x \in U$ where U is open in X then choose $i \in N$ such that $W(i,x) \subseteq U$. Let

$$V_2(i,z) \in G(i,b_1(i,x),\ m(i,x),\ b_2(i,x))$$

such that $x \in V_2(i,z)$. Thus $z \in W(b_2(i,z),x)$ and since $b_2(i,z) = b_2(i,x)$, it follows that

$$x \in W(b_2(i,x),x) \subseteq W(m(i,x),x)^o \subseteq V_1(i,x).$$

Hence $x \in W(b_1(i,x),z) \subseteq W(i,x)^o \subset U$. Since

$$V_2(i,z) \subseteq W(b_1(i,x),z) = W(b_1(i,z),z) \subseteq U$$

it follows that

$$st(x,G(i,b_1(i,x),m(i,x),b_2(i,x)) \subseteq U$$

and X is a quasi-developable space.

(iii) This is noted in the remark following Theorem 11 of [CR] and specifically proved in Theorem 3 of [CRRR].

Clearly imposing conditions A(1), A(2) and A(3) and a local neighborhood bases $\mathcal{W}$ for a T_1-space X is equivalent to the conditions in the Collins-Roscoe theorem. This represents the factorization of the Collins-Roscoe theorem.

Proof of Theorem 1. Since X has a countable local neighborhood basis satisfying A(1), A(2) and A(3) it is a quasi-developable space having closed sets G_δ-sets and, thus, is developable [Be]. It is also a collection-wise normal T_1-space and, hence, metrizable [Bi].

This factorization clearly shows the importance of the countable local neighborhood base being decreasing in the Collins-Roscoe Theorem. It also shows that a countable local neighborhood base that satisfies all the conditions of the Collins-Roscoe Theorem except being decreasing also implies a good deal of structure, i.e., quasi-developability, on the space.

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