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1. Introduction

Compact actions on S^2 were characterized by Kerekjarto in [K], where it is shown that the only regular homeomorphisms or S^2 are the (topological) rotations and reflections. Using this result, we observe (Theorem 2.3) that if H is an orientation preserving compact group action on a nondegenerate subcontinuum of S^2 such that H is extendable to a *compact* action G on S^2, then G and H are iseomorphic. (Note that this result does not necessarily hold when H is not orientation preserving. For example, take H to be the group generated by the reflection through the equator.) While Theorem 2.3 is an easy consequence of [K], it is remarkable in light of our Example 2.4, which explains the necessity of the hypothesis that the extension G be compact. (Or perhaps the theorem is natural, while the example is remarkable.)

Recall that a homeomorphism is *regular* iff its full family of iterates is equicontinuous.

2. Results

2.1 *Theorem. Let* X *be a nondegenerate continuum in* S^2 *(or* B^2*), and let* K *be an orientation preserving, compact group action on* S^2 *(*B^2*) such that each element of* K *is the identity on* X. *Then* K *is the identity.*

Proof. Kerekjarto [K] (see also [E]) has shown that the only orientation preserving, regular homeomorphisms of S^2 are the rotations. (See also [B,2] and [R].) Now every element of a compact group action is regular. Thus, since each non-identity rotation on S^2 has exactly a pair of points as its fixed point set, we see that K must be the identity.

If K acts on B^2, then it can be extended to an isomorphic action on S^2 (simply reflect the action), and by the above, it must again be the identity.

2.2 *Remark.* This theorem does not hold for S^3: Let $S^3 = T_1 \cup T_2$, where each T_i is a torus, and where T_1 and T_2 are identified by the homeomorphism g along their boundaries, where g identifies meridians on T_1 with longitudes on T_2. Then let h: $S^3 \twoheadrightarrow S^3$ be the period n homeomorphism that naturally extends a period n meridianal rotation on T_1. Then the fixed point set of h is the center circle, S, of T_1. Let $K = \{h, h^2, \cdots, h^n\}$, and let X = S. Then K is an orientation preserving extension of the identity on X, but K is not the identity.

2.3 *Theorem. Let* X *be a nondegenerate continuum in* S^2 *(*B^2*), and let* H *be a compact group action on* X*. Suppose that* H *can be extended to an orientation preserving compact group action* G *on* S^2 *(*B^2*). Then* G *and* H *are iseomorphic.*

Proof. Let f: G $\twoheadrightarrow$ H be the natural continuous homomorphism; that is, $f(g) = g|X$. We shall show that f is 1-1. Let $K = f^{-1}(e)$. Then K is a compact subgroup of G. By

Theorem 2.1, K must be the identity. It follows that f is 1-1, and therefore an iseomorphism.

2.4 *Example*. The following example shows that Theorem 2.3 does not hold in general; that is, restricted compact groups may not be iseomorphic to *any* group extension.

Let X be the outer boundary simple closed curve of the Sierpinski curve $Y \subseteq E^2 \subseteq S^2$, and let H be a circle group of homeomorphisms on X. The proof of the main theorem (3) of [W] shows that each element of H can be extended to a homeomorphism of Y onto itself. Choose such an extension for each $h \in H$, and let G be the group generated by these extensions. Then

(1) G is totally disconnected (see Theorem 1.2 of R. D. Anderson of [B,1]), but the orbit of a point in X is a simple closed curve.

(2) By (1), G' and H cannot be iseomorphic, for *any* group extension G' of H.

(3) G can be extended, in a natural way, to a group action $\tilde{G}$ on S^2, with $\tilde{G}$ iseomorphic to G, since each complementary domain of X in S^2 is a disk.

(4) G' cannot be compact, for any group extension G' of H. For if it were, by (3) and Theorem 2.3, G would be a circle group. But this contradicts (1).

(5) It follows from (4) and Theorem 2.3, that every compact subgroup of G is finite.

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