TOPOLOGY PROCEEDINGS
Volume 10, 1985
Pages 17–32
http://topology.auburn.edu/tp/

BALANCED Z-FILTERS

by

JOHN W. CARLSON

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

BALANCED Z-FILTERS

John W. Carlson

Introduction

A topological space Y is an extension of X if X is densely embedded in Y. We may assume that $\overline{X} = Y$ and then the nearness structure ξ generated by the extension Y is the collection of all $\mathcal{A} \subset \mathcal{P}(X)$ with $\cap \mathrm{cl}_Y \mathcal{A} \neq \phi$. The extension is a strict extension provided $\{\mathrm{cl}_Y A\colon A \subset X\}$ forms a base for the closed subsets of X. Bentley and Herrlich [2] have shown that if Y is a T_1 strict extension of X then Y is homeomorphic to (X^*, ξ^*), the completion of the nearness space (X, ξ).

For each $y \in Y$, $\mathcal{O}_y = \{Q \cap Y\colon y \in Q \in T_Y\}$ is an open filter on X. The collection $\{\mathcal{O}_y\colon y \in Y\}$ is called the filter trace of Y on X. For O open in X, let $O^* = \{y\colon O \in \mathcal{O}_y\}$. Then $\{O^*\colon O \in t\}$ is a base for the topology on Y, provided Y is a strict extension of X, Banaschewski [1].

The Stone-Čech compactification βX of a completely regular space X is a strict extension of X. For $p \in \beta X$, let $\mathcal{M}_p$ denote the z-ultrafilter on X that converges to p; $\mathcal{O}_p$ the trace filter of p; $\mathcal{V}_p$ the collection of co-zero sets in $\mathcal{O}_p$; $\mathcal{A}_p$ the collection of subsets of $A \subset X$ for which $p \in \mathrm{cl}_{\beta X} A$; and $\mathcal{M}_p^o = \{Z \in \mathcal{M}_p\colon \mathrm{cl}_{\beta X} Z$ is a nbhd of p in $\beta X\}$.

In this paper the relationship between the collections $\mathcal{M}_p$, $\mathcal{A}_p$, $\mathcal{O}_p$, $\mathcal{V}_p$, and $\mathcal{M}_p^o$ are discussed and it is shown that given any one of these collections the others can be determined in a natural way. These relationships are then used to provide a number of characterizations of the nearness structure on X induced by βX.

It is noted in [1] that the trace filters $\mathcal{O}_p$ generated by βX are the maximal completely regular open filters. It is shown here that $\mathcal{O}_p = \{O \in t_X\colon$ there exists $V \in \mathcal{V}_p$ with $V \subset O\}$ and that the $\mathcal{V}_p$ are the minimal prime co-zero filters on X. Moreover, it is shown that $\mathcal{V}_p = \{V\colon X-V \in Z(X)$ and $X-V \notin \mathcal{M}_p\}$ and that $\mathcal{M}_p = \{Z \in Z(X)\colon X-Z \notin \mathcal{V}_p\}$. Thus there exists a natural relationship between the minimal prime co-zero filters and the z-ultrafilters; a relationship that parallels the relationship between minimal prime open filters and closed ultrafilters [6].

A filter is simply the intersection of all the ultrafilters that contain it. A closed filter that is the intersection of the closed ultrafilters that contain it is called a balanced closed filter [6]. In this paper the concept of a balanced z-filter is introduced: a z-filter is called a balanced z-filter if it is the intersection of all the z-ultrafilters that contain it. It is then shown that there exists a natural one-to-one correspondence between the non-empty closed subsets of βX and the balanced z-filters on X. The paper is concluded by noting a few of the ways the topological properties of βX can be characterized in terms of balanced z-filters.

1. Preliminaries

For a completely regular space, let $Z(X)$ denote the collection of zero sets in X. V is called a co-zero set if $X-V \in Z(X)$. The collection of all co-zero sets on X will be denoted by co-$Z(X)$.

The construction βX will follow that found in Gillman and Jerrison [7]. Let βX be an index set that contains X and such that there exists a one-to-one correspondence between the points p of βX and the z-ultrafilters $\mathcal{M}_p$ on X. For $Z \in Z(X)$, the set $\overline{Z} = \{p \in \beta X: Z \in \mathcal{M}_p\}$. Then the collection $\{\overline{Z}: Z \in Z(X)\}$ is taken as a base for the closed sets in βX.

For zero sets Z and W in X it follows that:

(1) $p \in cl_{\beta X}Z$ if and only if $Z \in \mathcal{M}_p$,

(2) $cl_{\beta X}(Z \cap W) = cl_{\beta X}Z \cap cl_{\beta X}W$.

Let (Y,t) be a topological space and $\overline{X} = Y$. $t(X)$ will denote the subspace topology on X. For each $y \in Y$, set $\mathcal{O}_y = \{O \cap X: y \in O \in t\}$. Then $\{\mathcal{O}_y: y \in Y\}$ is called the filter trace of Y on X.

Let t(strict) be the topology on Y generated by the base $\{O^*: O \in t(X)\}$ where $O^* = \{y \in Y: O \in \mathcal{O}_y\}$. Let t(simple) be the topology on Y generated by the base $\{O \cup \{y\}: O \in \mathcal{O}_y, y \in Y\}$. Then t(strict) and t(simple) are such that Y with either of these topologies is an extension of $(X,t(X))$, called a strict extension, or simple extension of X, respectively. Note that

$$t(\text{strict}) \leq t \leq t(\text{simple}).$$

Moreover, a topology s on Y with the same filter trace as t, forms an extension of $(X,t(X))$ if and only if it satisfies the above inequality. (See Bànaschewski [1].)

Let X be a completely regular topological space. An open filter $\mathcal{O}$ is called a completely regular open filter if for each $O \in \mathcal{O}$ there exists $Q \in \mathcal{O}$ and a continuous function $f: X \to [0,1]$ with $f(Q) = 0$ and $f(X-O) = 1$.

It is known [1], that the trace filters of βX are the maximal completely regular open filters on X. Moreover, one can construct X by starting with the family of maximal completely regular open filters and construct βX using the strict extension topology.

Nearness spaces are obtained by axiomizing the concept of a collection of sets being "near"; in the same sense as a topological space is obtained by axiomizing the concept of a point being "near" a set. Herrlich, in [9], notes that each nearness structure can be obtained in a natural way by axiomizing the concept of collections of sets that are "far"; or the concept of a collection of sets that contain arbitrarily "small" sets, called merotopic structures; or by generalizing the concept of uniform covers. In this paper we will deal primarily with nearness structures and their corresponding uniform covers.

We will assume that the reader is basically familiar with the concept of a nearness space as defined by Herrlich in [8] and [9]. For a more recent treatment and survey, see Herrlich [10].

Definition 1.1. Let X be a set and μ a collection of covers of X, called uniform covers. Then (X,μ) is a nearness space provided:

(N1) $\mathcal{A} \in \mu$ and $\mathcal{A}$ refines $\mathcal{B}$ implies $\mathcal{B} \in \mu$.

(N2) $\{X\} \in \mu$ and $\phi \notin \mu$.

(N3) If $\mathcal{A} \in \mu$ and $\mathcal{B} \in \mu$ then $\mathcal{A} \wedge \mathcal{B} = \{A \cap B: A \in \mathcal{A}$ and $B \in \mathcal{B}\} \in \mu$.

(N4) $\mathcal{A} \in \mu$ implies $\{int(A): A \in \mathcal{A}\} \in \mu$. $(int(A) = \{x: X - \{x\}, A\} \in \mu)$.

For a given nearness space (X,μ) the collection of sets that are "near" is given by $\xi = \{\mathcal{A} \subset \mathcal{P}(X): \{X - A: A \in \mathcal{A}\} \notin \mu\}$. The micromeric collections are given by $\mathcal{A} \in \gamma$ if and only if $\{B \subset X: A \cap B \neq \phi$ for each $A \in \mathcal{A}\} \in \xi$ or equivalently, if for each $\mathcal{U} \in \mu$, there exists $U \in \mathcal{U}$ and $A \in \mathcal{A}$ with $A \subset U$. The closure operator generated by a nearness space is given by $cl_\xi(A) = \{x: \{\{x\},A\} \in \xi\}$. If we are primarily using these "near" collections we will denote the nearness space by (X,ξ). The underlying topology of a nearness space is always symmetric; that is, $x \in \overline{\{y\}}$ implies $y \in \overline{\{x\}}$.

Definition 1.2. Let (X,ξ) be a nearness space. The nearness space is called:

(1) topological provided $\mathcal{A} \in \xi$ implies $\cap\overline{\mathcal{A}} \neq \phi$.

(2) complete provided each ξ-cluster is fixed; that is $\cap\overline{\mathcal{A}} \neq \phi$ for each maximal element $\mathcal{A}$ in ξ.

(3) concrete provided each near collection is contained in some ξ-cluster.

(4) contigual provided $\mathcal{A} \notin \xi$ implies there exists a finite $\mathcal{B} \subset \mathcal{A}$ such that $\mathcal{B} \notin \xi$.

(5) totally bounded provided $\mathcal{A} \notin \xi$ implies there exists a finite $\mathcal{B} \subset \mathcal{A}$ such that $\cap\mathcal{B} = \phi$.

An extension Y of a space X is a space in which X is densely embedded. For notational convenience, we assume that $X \subset Y$. If Y is an extension of X then $\xi = \{\mathcal{A} \subset \mathcal{P}(X)\colon \cap \mathrm{cl}_Y \mathcal{A} \neq \phi\}$ is called the nearness structure induced on X by Y.

A brief description of the completion of a nearness space appears in [2] which we repeat here for the convenience of the reader. Let (X,ξ) be a nearness space and Y the set of all ξ-clusters. Set $X^* = X \cup Y$. For $A \subset X$, define $\mathrm{cl}(A) = \mathrm{cl}_\xi A \cup \{y \in Y\colon A \in y\}$. A nearness structure ξ^* is defined on X^* as follows: $\mathcal{B} \in \xi^*$ provided $\{A \subset X\colon$ there exists $B \in \mathcal{B}$ with $B \subset \mathrm{cl}(A)\} \in \xi$. (X^*,ξ^*) is a complete nearness space with $\mathrm{cl}_{\xi^*}X = X^*$. Also; for $A \subset X$, $\mathrm{cl}_{\xi^*}(A) = \mathrm{cl}(A)$.

The following two important theorems are due to Bentley and Herrlich [2].

Theorem A. *For any* T_1 *nearness space* (X,ξ) *the following conditions are equivalent:*

(1) ξ *is a nearness structure induced on* X *by a strict extension.*

(2) *The completion* X^* *of* X *is topological.*

(3) ξ *is concrete.*

Theorem B. *Strict extensions are equivalent if and only if they induce the same nearness structure.*

2. Minimal Prime Co-Zero Filters

If $\mathcal{F}$ is an ultrafilter on X then $\mathcal{F} = \{F \subset X\colon X - F \notin \mathcal{F}\}$. Recall that a filter is an ultrafilter if and only if it is a prime filter; a result that does not carry for open ultrafilters, closed ultrafilters nor z-ultrafilters.

Let $\mathcal{F}$ be a closed filter and $\mathcal{G} = \{0 \in t: X - 0 \notin \mathcal{F}\}$. Then:

(1) $\mathcal{F}$ is a closed ultrafilter if and only if $\mathcal{G}$ is a minimal prime open filter.

(2) $\mathcal{G}$ is an open ultrafilter if and only if $\mathcal{F}$ is a minimal prime closed filter.

The first statement was useful in discussing Pervin nearness structures [6]. The trace filters of the completion of an ultrafilter generated nearness space are the corresponding minimal prime open filters [4]. The second statement above follows in a natural way.

The same construction can be employed with z-ultrafilters; and this construction yields a minimal prime co-zero filter. The trace filters of X are, Banaschewski [1], the maximal completely regular open filters on X, and as we shall see in section 3, these are the open filters generated by the minimal prime co-zero filters.

Let X be a completely regular space. We will let co-Z(X) denote the collection of co-zero sets on X.

Theorem 2.1. *Let* $\mathcal{M}_p$ *be a* z-*ultrafilter on a completely regular space* X. *Set* $\mathcal{V}_p = \{V \in \text{co-Z}(X): X - V \notin \mathcal{M}_p\}$. *Then* $\mathcal{V}_p$ *is a minimal prime co-zero filter on* X.

Proof. Easily $\phi \notin \mathcal{V}_p$ and if $V \in \mathcal{V}_p$ and $V \subset W \in \text{co-Z}(X)$ then $X - W \subset X - V \notin \mathcal{M}_p$ and thus $X - W \notin \mathcal{M}_p$. Hence $W \in \mathcal{V}_p$. Let $V_1 \in \mathcal{V}_p$ and $V_2 \in \mathcal{V}_p$. Then $X - V_1 \notin \mathcal{M}_p$ and $X - V_2 \notin \mathcal{M}_p$ and thus $X - (V_1 \cap V_2) = (X - V_1) \cup (X - V_2) \notin \mathcal{M}_p$.

Therefore, $V_1 \cap V_2 \in \mathcal{V}_p$ and $\mathcal{V}_p$ is a co-zero filter.

To see that $\mathcal{V}_p$ is prime let $V_1 \cup V_2 \in \mathcal{V}_p$. Then $X - (V_1 \cup V_2) \notin \mathcal{M}_p$. If $V_1 \notin \mathcal{V}_p$ and $V_2 \notin \mathcal{V}_p$ it follows that $X - V_1 \in \mathcal{M}_p$ and $X - V_2 \in \mathcal{M}_p$ and thus $X - (V_1 \cup V_2) = (X - V_1) \cap (X - V_2) \in \mathcal{M}_p$, a contradiction. Thus, $\mathcal{V}_p$ is a prime co-zero filter.

Suppose $\mathcal{W}$ is a prime co-zero filter with $\mathcal{W} \subset \mathcal{V}_p$. Then, $\mathcal{N} = \{Z \in Z(X): X - Z \notin \mathcal{W}\}$ is a prime z-filter. Moreover, $\mathcal{M}_p \subset \mathcal{N}$ and since $\mathcal{M}_p$ is a z-ultrafilter $\mathcal{M}_p = \mathcal{N}$. Thus, $\mathcal{W} = \mathcal{V}_p$ and $\mathcal{V}_p$ is a minimal prime co-zero filter on X.

Essentially the same argument yields the following theorem.

Theorem 2.2. *Let* X *be a completely regular topological space and let* $\mathcal{V}$ *be a minimal prime co-zero filter. Set*

$$\mathcal{M} = \{Z \in Z(X): X - Z \notin \mathcal{V}\}.$$

Then, $\mathcal{M}$ *is a* z-*ultrafilter.*

3. Characterizations of the Nearness Structure Induced By βX

Let X be a completely regular space and βX the Stone-Čech compactification of X. The collections $\mathcal{M}_p$, $\mathcal{M}_p^o$, $\mathcal{A}_p$, $\mathcal{O}_p$, and $\mathcal{V}_p$ are induced in a natural way. Specifically, if $p \in \beta X$, then:

$$\mathcal{M}_p = \{Z: p \in cl_{\beta X} Z\}$$
$$\mathcal{M}_p^o = \{Z: cl_{\beta X} Z \text{ is a nbhd of } p \text{ in } \beta X\}$$
$$\mathcal{A}_p = \{A \subset X: p \in cl_{\beta X} A\}$$
$$\mathcal{O}_p = \{0 \in t: p \notin cl_{\beta X}(X - 0)\}$$
$$\mathcal{V}_p = \{V \in \text{co-}Z(X): p \notin cl_{\beta X}(X - V)\}$$

As noted in section 1, one can construct βX by starting either with the family of z-ultrafilters or the family of maximal completely regular open filters. In this section we show that given any one of the above collections we can determine the other four in a natural way. Thus, it follows that if we have any one of these collections we can recover βX. The relationships between these collections are stated in the following theorem. Statement 3-A can be found in Gillman and Jerison [7] and is included here for completeness. (Note: Our $\mathcal{M}_p^o$ corresponds to $Z[0^p]$ found in [7].)

Theorem 3.1. *Let* (X,t) *be a completely regular space and* $p \in \beta X$.

(1) *Let* $\mathcal{M}_p$ *be a* z*-ultrafilter. Then:*

(A) $\mathcal{O}_p = \{0 \in t$: *There exists* $Z \notin \mathcal{M}_p$ *with* $Z \supset X - 0\}$

(B) $\mathcal{A}_p = \{A \subset X$: *If* $Z \supset A$ *then* $Z \in \mathcal{M}_p\}$

(C) $\mathcal{M}_p^o = \{Z \in \mathcal{M}_p$: *There exists* $Z' \notin \mathcal{M}_p$ *with* $Z \cup Z' = X\}$

(D) $\mathcal{V}_p = \{V \in \text{co-}Z(X): X - V \notin \mathcal{M}_p\}$

(2) *Let* $\mathcal{O}_p$ *be a maximal completely regular open filter. Then:*

(A) $\mathcal{M}_p = \{Z \in Z(X): 0 \cap Z \neq \phi$ *for each* $0 \in \mathcal{O}_p\}$

(B) $\mathcal{A}_p = \{A \subset X: A \cap 0 \neq \phi$ *for each* $0 \in \mathcal{O}_p\}$

(C) $\mathcal{M}_p^o = \{Z \in Z(X)$: *There exists* $0 \in \mathcal{O}_p$ *with* $Z \supset 0\}$

(D) $\mathcal{V}_p = \{V \in \text{co-}Z(X): V \in \mathcal{O}_p\}$

(3) *Let* $\mathcal{M}_p^o$ *be given. Then:*

(A) $\mathcal{M}_p = \{Z' \in Z(X): Z \cap Z' \neq \phi$ *for each* $Z \in \mathcal{M}_p^o\}$

(B) $\mathcal{A}_p = \{A \subset X: A \cap Z \neq \phi$ *for each* $Z \in \mathcal{M}_p^o\}$

(C) $\mathcal{O}_p = \{0 \in t$: *There exists* $Z \in \mathcal{M}_p^o$ *with* $0 \supset Z\}$

(D) $\mathcal{V}_p = \{V \in \text{co-}Z(X)$: *There exists* $Z \in \mathcal{M}_p^o$ *with* $V \supset Z\}$

(4) *Let* $\mathcal{A}_p$ *be a* ξ_X*-cluster. Then:*

(A) $\mathcal{M}_p = \{Z \in Z(X): Z \in \mathcal{A}_p\}$

(B) $\mathcal{O}_p = \{0 \in t: X - 0 \notin \mathcal{A}_p\}$

(C) $\mathcal{M}_p^o = \{Z \in Z(X): X - \mathrm{Int}(Z) \notin \mathcal{A}_p\}$

(D) $\mathcal{V}_p = \{V \in \text{co-}Z(X): X - V \notin \mathcal{A}_p\}$

(5) *Let* $\mathcal{V}_p$ *be a minimal prime co-zero filter. Then:*

(A) $\mathcal{O}_p = \{0 \in t$: *There exists* $V \in \mathcal{V}_p$ *with* $0 \supset V\}$

(B) $\mathcal{A}_p = \{A \subset X: A \cap V \neq \phi$ *for each* $V \in \mathcal{V}_p\}$

(C) $\mathcal{M}_p^o = \{Z \in Z(X)$: *There exists* $V \in \mathcal{V}_p$ *with* $Z \supset V\}$

(D) $\mathcal{M}_p = \{Z \in Z(X): X - Z \notin \mathcal{V}_p\}$

The relationships between these various collections allow us to find a variety of characterizations for the nearness structure induced by the Stone-Čech compactification.

Let X be a completely regular space. Then the nearness structure on X induced by βX is $\xi_{\beta X} = \{\mathcal{A} \subset \mathcal{P}(X): \cap \mathrm{cl}_{\beta X}\mathcal{A} \neq \phi\}$.

Herrlich, in [9], has characterized $\xi_{\beta X}$. His characterization is included in Theorem 3.2, statement 1-D, for completeness. The reader is referred to [9] for his construction of the contigual reflector, ξ_c, and the uniform reflector, ξ_u.

In Theorem 3.2, we provide several characterizations of the nearness structure induced by βX; some stated in terms of the near collections, others in terms of the cover structure or the micromeric collections.

Theorem 3.2. *Let* X *be a completely regular topological space. Let* $\xi_{\beta X}$ *denote the nearness structure induced on* X *by* βX *and let* $\mu_{\beta X}$ *and* $\gamma_{\beta X}$ *denote the corresponding collection of uniform covers and micromeric collections, respectively. Then:*

(1) $\xi_{\beta X}$.

(A) $\xi_{\beta X} = \{\mathcal{A} \subset \mathcal{P}(X)$: *There exists a* z-*ultrafilter* $\mathcal{M}_p$ *such that if* $Z \in Z(X)$ *and* $Z \supset A$ *for some* $A \in \mathcal{A}$ *then* $Z \in \mathcal{M}_p\}$

(B) $\xi_{\beta X} = \{\mathcal{A} \subset \mathcal{P}(X)$: *There exists* $p \in \beta X$ *with* $\mathcal{A} \subset \mathcal{A}_p\}$

(C) $\xi_{\beta X} = \{\mathcal{A} \subset \mathcal{P}(X)$: *There exists* $p \in \beta X$ *with* $0 \cap A \neq \phi$ *for each* $0 \in \mathcal{O}_p$ *and* $A \in \mathcal{A}\}$

(D) (Herrlich [9]) $\xi_{\beta X} = \xi_{tcu} = \xi_{tuc}$

(2) $\mu_{\beta X}$.

(A) $\mu_{\beta X} = \{\mathcal{U} \subset \mathcal{P}(X)$: *There exists a sequence of finite open covers* $\mathcal{O}_i$ *such that* $\cdots <^* \mathcal{O}_{k+1} <^* \mathcal{O}_k <^* \cdots <^* \mathcal{O}_1 <^* \mathcal{U}\}$

(B) $\mu_{\beta X} = \{\mathcal{U} \subset \mathcal{P}(X)$: *For each* z-*ultrafilter* $\mathcal{M}_p$ *there exists* $U \in \mathcal{U}$ *and a zero set* $Z \notin \mathcal{M}_p$ *with* $Z \supset X - U\}$

(C) $\mu_{\beta X} = \{\mathcal{U} \subset \mathcal{P}(X)$: *For each* z-*ultrafilter* $\mathcal{M}_p$ *there exists* $U \in \mathcal{U}$ *and a zero set* $Z \notin \mathcal{M}_p$ *with* $X - Z \subset U\}$

(D) $\mu_{\beta X} = \{\mathcal{U} \subset \mathcal{P}(X)$: *For each minimal prime co-zero filter* $\mathcal{V}_p$ *there exists* $U \in \mathcal{U}$ *and* $V \in \mathcal{V}_p$ *with* $V \subset U\}$

(3) $\gamma_{\beta X}$.

(A) $\gamma_{\beta X} = \{\mathcal{S} \subset \mathcal{P}(X)$: *There exists a maximal completely regular open filter* $\mathcal{O}_p$ *that corefines* $\mathcal{S}\}$

(B) $\gamma_{\beta X} = \{\mathcal{S} \subset \mathcal{P}(X)$: *There exists a minimal prime co-zero filter* $\mathcal{V}_p$ *that corefines* $\mathcal{S}\}$

4. Balanced Z-Filters

Definition 4.1. Let $\mathcal{F}$ be a zero filter on X. $\mathcal{F}$ is called a balanced zero filter provided $\mathcal{F}$ is the intersection of all the z-ultrafilters that contain it. That is; $\mathcal{F} = \cap\{\mathcal{M}_p: \mathcal{F} \subset \mathcal{M}_p\}$.

Let the sec operator be with respect to zero sets; that is, $\sec(\mathcal{F}) = \{Z \in Z(X): Z \cap Z' \neq \phi$ for each $Z' \in \mathcal{F}\}$. Then a zero filter $\mathcal{F}$ is balanced if and only if $\mathcal{F} = \sec^2(\mathcal{F})$. A filter on a set is simply the intersection of all the ultrafilters that contain it. This need not be the case in general for open filters, closed filters, or zero filters. (For example, a closed filter is not necessarily the intersection of all the closed ultrafilters that contain it; if it is, it is called a balanced closed filter.)

Balanced near collections and balanced closed filters were discussed in [6]. In that paper it was shown that, for a normal space X, there exists a one-to-one correspondence between the nonempty closed sets in βX and the balanced closed filters on X. As a special case it was shown that there exists a one-to-one correspondence between the nonempty closed subsets of βN and the filters on N. These results are extended in this paper using the concept of a balanced zero filter.

Theorem 4.1. *Let* G *be a nonempty closed set in* βX. *Set* $\mathcal{F} = \{Z \in Z(X): G \subset cl_{\beta X}Z\}$. *Then:*

(1) $G = \cap\{cl_{\beta X}Z: Z \in \mathcal{F}\}$.

(2) $\mathcal{F}$ *is a balanced* z-*filter.*

Proof. (1) follows since $\{cl_{\beta X}Z: Z \in Z(X)\}$ is a base for the closed sets in βX. (A parallel argument is that βX is a strict extension of X and the zero sets in X form a base for the closed sets in X.)

(2) Easily $\phi \notin \mathcal{F}$ and if $Z \in \mathcal{F}$ with $Z \subset Z'$ then $Z' \in \mathcal{F}$. If $Z_1 \in \mathcal{F}$ and $Z_2 \in \mathcal{F}$ then since $cl_{\beta X}Z_1 \cap cl_{\beta X}Z_2 = cl_{\beta X}(Z_1 \cap Z_2)$, we have that $G \subset cl_{\beta X}(Z_1 \cap Z_2)$ and $Z_1 \cap Z_2 \in \mathcal{F}$. Thus, $\mathcal{F}$ is a z-filter.

To see that $\mathcal{F}$ is balanced, it suffices to let $Z \in \mathcal{M}_p$ for each $\mathcal{M}_p$ containing $\mathcal{F}$ and show that $Z \in \mathcal{F}$. Let $Z \in \cap\{\mathcal{M}_p: \mathcal{F} \subset \mathcal{M}_p\}$. Then $p \in cl_{\beta X}Z$ for each $\mathcal{M}_p$ containing $\mathcal{F}$.

Let $q \in G$. Then $q \in cl_{\beta X}Z$ for each $Z \in \mathcal{F}$. Hence $\mathcal{F} \subset \mathcal{M}_q$. Thus, if $Z \in \cap\{\mathcal{M}_p: \mathcal{F} \subset \mathcal{M}_p\}$ we have that $G \subset cl_{\beta X}Z$. Therefore, $Z \in \mathcal{F}$, and $\mathcal{F}$ is a balanced z-filter.

The following theorem now follows immediately.

Theorem 4.2. *Let* X *be a completely regular topological space. Then there exists a one-to-one correspondence between the nonempty closed sets in* βX *and the balanced z-filters on* X. *The correspondnece is given by:* $G \leftrightarrow \mathcal{F} = \{Z: G \subset cl_{\beta X}Z\}$.

Corollary 4.3. *Let* 0 *be an open set in* βX *and* $0 \neq \beta X$. *Then there exists a balanced* z-*filter* $\mathcal{F}$ *such that* $0 = \{p \in \beta X: \mathcal{F} \not\subset \mathcal{M}_p\}$.

Let N denote the natural numbers with the discrete topology. Since every subset of N is a zero set, and hence each filter on N is a z-filter, the following result obtained in Carlson [6] follows as a natural consequence of Theorem 4.2.

Corollary 4.4. *There exists a natural one-to-one correspondence between the nonempty closed subsets of* βN *and the filters on* N.

Corollary 4.5. *Let* X *be a completely regular space. Then there exists a one-to-one correspondence between the zero sets in* βX *and the balanced* z*-filters on* X *with a countable base.*

Proof. It is known, (6E) in [7], that each nonempty zero set in βX is the countable intersection of sets of the form $\mathrm{cl}_{\beta X} Z$, where $Z \in Z(X)$. The result now follows from Theorem 4.2.

Theorem 4.6. *Let* G *be a nontrivial open-closed subset of* βX. *Then there exists balanced* z*-filters* $\mathcal{F}$ *and* $\mathcal{G}$ *such that each* z*-ultrafilter contains one and only one of these filters.*

Lemma 4.7. *Let* X *be a completely regular space with* F_1 *and* F_2 *nonempty closed subsets of* βX. *For* $i = 1,2$; *let* $O_i = \beta X - F_i$ *and* $\mathcal{F}_i = \{Z \in Z(X): F_i \subset \mathrm{cl}_{\beta X} Z\}$. *The following statements are equivalent.*

(1) $F_1 \subset F_2$

(2) $O_2 \subset O_1$

(3) $\mathcal{F}_2 \subset \mathcal{F}_1$.

Proof. By Theorem 5.2, $F_i = \{p \in \beta X: \mathcal{F}_i \subset \mathcal{M}_p\}$ for $i = 1,2$. Suppose $F_1 \subset F_2$. Let $Z \in \mathcal{F}_2$. Then $F_2 \subset \mathrm{cl}_{\beta X} Z$

and thus $F_1 \subset cl_{\beta X}Z$ and $Z \in \mathcal{F}_1$. Suppose $\mathcal{F}_2 \subset \mathcal{F}_1$. Let $p \in F_1$, then $\mathcal{F}_1 \subset \mathcal{M}_p$ and thus $\mathcal{F}_2 \subset \mathcal{M}_p$ and $p \in F_2$.

Definition 4.2. A point p in a topological space X is called a P-point if for each countable collection $\{0_i\}$ of open sets containing p there exists an open set 0 with $p \in 0 \subset \cap 0_i$.

Given a z-ultrafilter $\mathcal{M}_p$ on a completely regular space we now characterize when p is a P-point in βX in terms of balanced z-filters.

Theorem 4.8. *Let* X *be a completely regular topological space and* $\mathcal{M}_p$ *a* z*-ultrafilter on* X. p *is a* P*-point in* βX *if and only if for each countable collection* $\{\mathcal{F}_i\}$ *of balanced* z*-filters with* $\mathcal{F}_i \not\subset \mathcal{M}_p$ *there exists a balanced* z*-filter* $\mathcal{F}$ *such that* $\mathcal{F} \not\subset \mathcal{M}_p$ *and for each* i, $\mathcal{F} \subset \mathcal{F}_i$.

Proof. Let p be a P-point in βX, and $\{\mathcal{F}_i: i \in N\}$ a countable collection of balanced z-filters with $\mathcal{F}_i \not\subset \mathcal{M}_p$. Set $F_i = \cap cl_{\beta X}\mathcal{F}_i$. Then F_i is a closed set with $p \notin F_i$. Then $0_i = \beta X - F_i$ is an open set containing p for each $i \in N$. Since p is a P-point in βX, there exists an open set 0 in βX with $p \in 0 \subset \cap 0_i$. Let $F = \beta X - 0$ and $\mathcal{F}$ the corresponding balanced z-filter on X. Then for each $i \in N$, $F \supset F_i$ and, by Lemma 4.7, $\mathcal{F} \subset \mathcal{F}_i$. Easily, $\mathcal{F} \not\subset \mathcal{M}_p$. The proof in the other direction follows in a similar manner.

Let $\{\mathcal{F}_i\}$ be a countable collection of filters on N. Then $\cap\mathcal{F}_i$ is a filter on N and we have the following corollary.

Corollary 4.9. *Let* N *denote the natural numbers with the discrete topology. Let* $\mathcal{M}$ *be an ultrafilter on* N. *Then* $\mathcal{M}$ *is a* P*-point in* βN *if and only if for each countable family of filters* $\{\mathcal{F}_i\}$ *on* N, *with each* $\mathcal{F}_i \not\subset \mathcal{M}$, *then* $\cap\mathcal{F}_i \not\subset \mathcal{M}$.

References

1. B. Banaschewski, *Extensions of topological spaces*, Can. Math. Bul. 7 (1964), 1-22.
2. H. L. Bentley and H. Herrlich, *Extensions of topological spaces*, Topological Proc., Memphis State Univ. Conference, Marcel-Dekker, New York (1976), 120-184.
3. J. W. Carlson, *Topological properties in nearness spaces*, Gen. Top. Appl. 8 (1978), 111-118.
4. ________, *Prime extensions and nearness structures*, Commentationes Math. Univ. Carolinae 21 (1980), 663-677.
5. ________, *Nearness and quasi-uniform structures*, Top. Appl.]5 (1983), 19-28.
6. ________, *Pervin nearness spaces*, Top. Proc. 9 (1984),7-30.
7. L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand Reinhold Press, Princeton, New Jersey (1960).
8. H. Herrlich, *A concept of nearness*, Gen. Top. Appl. 5 (1974), 191-212.
9. ________, *Topological structures*, Mathematical Centre Tracts 52, Amsterdam (1974).
10. ________, *Categorical topology 1971-1981*, Proc. Fifth Prague Topol. Symp. 1981, Heldermann Verlag Berlin (1982), 279-383.

Emporia State University

Emporia, Kansas 66801