A TOPOLOGICAL PROOF OF PAROVIČENKO’S CHARACTERIZATION OF $\beta \mathbb{N} \setminus \mathbb{N}$

by

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The following properties of the remainder $P = \beta N - N$ are well-known:

1. $P$ is a zero-dimensional compact space without isolated points.
2. Every two disjoint open $F_\sigma$-sets in $P$ have disjoint closures (i.e., $P$ is an $F$-space).
3. Every non-empty $G_\delta$-set in $P$ has a non-empty interior.

As shown by the four propositions below, these are in fact properties of a larger class of remainders.

Proposition 1. For each strongly zero-dimensional space $X$ the remainder $\beta X - X$ is zero-dimensional.

Proposition 2. For each $\sigma$-compact space $X$ the remainder $\beta X - X$ has no isolated points.

Proposition 3. ([GH]). For each locally compact $\sigma$-compact space $X$ the remainder $\beta X - X$ is an $F$-space.

Proposition 4. ([FG]). For each locally compact realcompact space $X$ every non-empty $G_\delta$-set in $\beta X - X$ has a non-empty interior.

We shall call a space $P$ a Parovićenko space, if $P$ has properties (1)-(3). Since every infinite compact

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F-space contains a copy of \( \aleph N \) ([GJ], 14N.5), every Parovičenko space has weight \( \geq c \).

In his 1963 paper [P] Parovičenko established the following two theorems:

First Parovičenko Theorem. Every compact space of weight \( \leq \aleph_1 \) is a continuous image of \( \aleph N - N \).

Second Parovičenko Theorem. The Continuum Hypothesis implies that every Parovičenko space of weight \( c \) is homeomorphic to \( \aleph N - N \).

The original proofs were in Boolean algebraic language, and no topological proofs were available until 1980, when Błaszczyk and Szymański presented in [BS] a proof of the First Parovičenko Theorem using the inverse systems technique. Developing their ideas, we shall present here a topological proof of the Second Parovičenko Theorem. Our terminology and notations follow [E].

We start with two characterizations of Parovičenko spaces:

**Lemma.** For every compact space \( P \) the following conditions are equivalent.

(i) \( P \) is a Parovičenko space.

(ii) For every continuous mapping \( f \) of \( P \) onto a compact metrizable space \( X \) and every pair \( F_1, F_2 \) of closed subsets of \( X \) such that \( F_1 \cup F_2 = X \) there exists an open-and-closed set \( U \subseteq P \) such that \( f(U) = F_1 \) and \( f(P - U) = F_2 \).

(iii) For every continuous mapping \( f \) of \( P \) onto a compact metrizable space \( X \) and every continuous mapping \( g \) of a
compact metrizable space $Y$ onto $X$ there exists a continuous mapping $h$ of $P$ onto $Y$ such that $gh = f$.

Let us precede the proof of our Lemma by brief comments on conditions (ii) and (iii). Condition (iii) first appeared in [N], where it was proved under CH that $\beta N - N$ satisfies (iii) and that (iii) topologically characterizes $\beta N - N$ in the class of all compact spaces of weight $\aleph_1$ (as a matter of fact, there was one more requirement in the characterization of $\beta N - N$ given in [N], viz. that every non-empty compact metric space $M$ is a continuous image of $\beta N - N$, but this follows directly from (iii) by letting $X = \{0\}$ and $Y = M$). Condition (ii) was introduced in [BS], where it was proved that (i) and (ii) are equivalent for every compact space $P$ and that (iii) implies (ii) (in that paper (iii) is misstated: the requirement that $h(P) = Y$ is missing and without this requirement one cannot show in the proof of the implication (iii) $\Rightarrow$ (ii) that $f(U) = F_1$ and $f(P - U) = F_2$).

Proof of the Lemma. It remains to prove that (ii) implies (iii). Since every compact metrizable space $Y$ is a continuous image of the Cantor set $D_{\aleph_0}$ (see [E], Problem 4.5.9(b)), it suffices to prove (iii) under the additional assumption that $Y = D_{\aleph_0} = \Pi_{i=1}^{\aleph_0} D_i$, where $D_i = D = \{0,1\}$ for $i = 1, 2, \ldots$. For every finite sequence $i_1, i_2, \ldots, i_k$ of zeros and ones the set $F_{i_1 i_2 \ldots i_k} = \{i_1, i_2, \ldots, i_k\} \times \Pi_{i=k+1}^{\aleph_0} D_i$ is open-and-closed in $D_{\aleph_0}$. Applying (ii) we can define inductively open-and-closed subsets $U_{i_1 i_2 \ldots i_k}$ of $P$ such that
For each $x \in P$ there is exactly one infinite sequence $i_1, i_2, \ldots$ such that $x \in U_{i_1 i_2 \cdots i_k}$ for $k = 1, 2, \ldots$, and the corresponding intersection $\bigcap_{k=1}^{\infty} F_{i_1 i_2 \cdots i_k}$ consists of a single point, so that by letting

$$h(x) = \bigcap_{k=1}^{\infty} F_{i_1 i_2 \cdots i_k} \quad \text{for } x \in \bigcap_{k=1}^{\infty} U_{i_1 i_2 \cdots i_k}$$

we define a mapping $h: P \to D$. Since the sets $F_{i_1}, F_{i_1 i_2}, \ldots$ are closed in the compact space $D$ and form a decreasing sequence, we have $g(\bigcap_{k=1}^{\infty} F_{i_1 i_2 \cdots i_k}) = \bigcap_{k=1}^{\infty} g(F_{i_1 i_2 \cdots i_k})$, so that $f(x) = \bigcap_{k=1}^{\infty} f(U_{i_1 i_2 \cdots i_k}) = \bigcap_{k=1}^{\infty} g(F_{i_1 i_2 \cdots i_k}) = g(\bigcap_{k=1}^{\infty} F_{i_1 i_2 \cdots i_k}) = g(h(x))$, and thus $gh = f$. One easily checks that

$$h^{-1}(F_{i_1 i_2 \cdots i_k}) = U_{i_1 i_2 \cdots i_k};$$

the family of all the sets $F_{i_1 i_2 \cdots i_k}$ being a base for $D$, this implies that $h$ is continuous and maps $P$ onto $D$.

**Proof of the Second Parovičenko Theorem.** It suffices to prove that any Parovičenko spaces $P, X$ of weight $\aleph_1$ are homeomorphic. Since $X$ is embeddable in $I$, one can assume that $X = \lim_{\alpha < \omega_1} \{X_{\alpha}, \pi_{\alpha}^{\alpha}, \alpha < \omega_1\}$, where $X_{\alpha}$ are compact metrizable spaces, projections $\pi_{\alpha}: X \to X_{\alpha}$ are mappings onto, and for each limit number $\lambda < \omega_1$ we have $\lim_{\alpha < \lambda} X_{\alpha}, \pi_{\alpha}^{\alpha}, \alpha < \lambda = X_{\lambda}$ (see [E], Proposition 2.5.6). Let $\{V_{\alpha}, \alpha \in A\}$, where
A is the set of all non-limit countable ordinal numbers, be a base for the space P consisting of open-and-closed sets with $V_1 = P$.

Applying transfinite induction we shall define for each $\alpha < \omega_1$ a countable ordinal number $\phi(\alpha) > \alpha$ and a continuous mapping $f_\alpha$ of $P$ onto $X_{\phi(\alpha)}$ such that

1. $\phi(\beta) < \phi(\alpha)$ and $\pi_{\phi(\alpha)} f_\beta = f_\beta$ for $\beta < \alpha$ and
2. $f_\alpha(V_\alpha) \cap f_\alpha(P - V_\alpha) = \emptyset$ if $\alpha \in A$.

Let $\phi(1) = 1$ and $f_1$ be an arbitrary continuous mapping of $P$ onto $X_1$; conditions (1) and (2) are satisfied for $\alpha = 1$.

Assume that $\phi(\alpha)$ and $f_\alpha$ are defined for $\alpha < \gamma$ and satisfy conditions (1) and (2).

If $\gamma$ is a limit number, we define $\phi(\gamma) = \sup\{\phi(\alpha): \alpha < \gamma\}$ and $f_\gamma = \lim_{\alpha < \gamma} f_\alpha$. Condition (1) is satisfied for $\alpha = \gamma$ and, by Corollary 3.1.16 in [E], $f_\gamma$ maps $P$ onto $\lim_{\alpha < \gamma} X_{\phi(\alpha)} = X_{\phi(\gamma)}$.

If $\gamma = \delta + 1$, the ordinal number $\phi(\delta)$ and the mapping $f_\delta$ are already defined, and the sets $f_\delta(V_\gamma)$, $f_\delta(P - V_\gamma)$ are closed and cover the space $X_{\phi(\delta)}$. Since $X$ is a Parovičenko space, there exists by (ii) an open-and-closed set $U \subset X$ such that $\pi_{\phi(\delta)}(U) = f_\delta(V_\gamma)$ and $\pi_{\phi(\delta)}(X - U) = f_\delta(P - V_\gamma)$.

The limit of an inverse system of non-empty compact spaces being non-empty, there exists a countable ordinal number $\phi(\gamma)$, larger than both $\gamma$ and $\phi(\delta)$, such that $\pi_{\phi(\gamma)}(U) \cap \pi_{\phi(\gamma)}(X - U) = \emptyset$. Now, since every non-empty open-and-closed subspace of a Parovičenko space is a Parovičenko space, there exist by (iii) continuous mappings
\( f'_{\gamma} \) of \( V_{\gamma} \) onto \( \pi_{\phi}(\gamma)(U) \) and \( f''_{\gamma} \) of \( P - V_{\gamma} \) onto \( \pi_{\phi}(\gamma)(X - U) \) such that
\[
\pi_{\phi}(\gamma)f'_{\gamma}(x) = f_{\delta}(x) \text{ for } x \in V_{\gamma} \text{ and }
\pi_{\phi}(\gamma)f''_{\gamma}(x) = f_{\delta}(x) \text{ for } x \in P - V_{\gamma}.
\]
By letting
\[
f_{\gamma}(x) = \begin{cases} 
  f'_{\gamma}(x), & \text{if } x \in V_{\gamma} \\
  f''_{\gamma}(x), & \text{if } x \in P - V_{\gamma}
\end{cases}
\]
we define a continuous mapping \( f_{\gamma} \) of \( P \) onto \( X_{\phi(\gamma)} \) such that (1) and (2) are satisfied for \( \alpha = \gamma \).

The limit mapping \( f = \lim_{\alpha < \omega_1} f_{\alpha} \) maps \( P \) onto \( \lim_{\alpha < \omega_1} X_{\phi(\alpha)} = X \). To show that \( f \) is a homeomorphism, it suffices to observe that by (2) \( f \) is a one-to-one mapping.

Let us add that a proof of the First Parovičenko Theorem, in principle identical with the one given in [BS], can be obtained by obvious simplifications in the above proof. It should be also added that, as established in [DM], the assumption that every Parovičenko space of weight \( c \) is homeomorphic to the remainder \( \beta N - N \) implies the Continuum Hypothesis.

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References


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