CONCERNING THE EXTENSION OF CONNECTIVITY FUNCTIONS

by

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In his classic paper, Stallings [7] asked if a connectivity function $I \to I$ could always be extended to a connectivity function $I^2 \to I$ when $I$ is considered embedded in $I^2$ as $I \times 0$. Several authors answered this negatively by giving examples of connectivity functions $I \to I$ which are not almost continuous, [1], [6]. In [7] Stallings proved that an almost continuous function $I \to I$ is a connectivity function and, curiously enough, a connectivity function $I^2 \to I$ is an almost continuous function. Later it was shown by Kellum [4] that an almost continuous function $I \to I$ can be extended to an almost continuous function $I^2 \to I$. This naturally leaves the question "can an almost continuous function $I \to I$ be extended to a connectivity function $I^2 \to I"?" Theorem 2 of this paper together with the first example of [2] shows that this is not the case.

For simplicity no distinction will be made between points of $I \times 0$ and $I$. Also, $B(y,r)$ denotes an open ball about $y$ with radius $r$ where $d$ is the usual distance function.

Definition 1. A function $f: X \to Y$ between spaces $X$ and $Y$ is said to be almost continuous if each open set containing the graph of $f$ also contains the graph of a continuous function with the same domain. The function $f$ is said to be a connectivity function if for each connected subset
C of X the graph of f restricted to C, denoted by \( f|_C \), is a connected subset of \( X \times Y \). The function f is said to be a Darboux function if \( f(C) \) is connected whenever C is a connected subset of X.

**Definition 2.** A function \( f: I \to I \) has the Cantor Intermediate Value Property (CIVP) if for any Cantor set \( K \) in the interval \( (f(x), f(y)) \) the interval \( (x, y) \) or \( (y, x) \) contains a Cantor set \( C \) such that \( f(C) \subseteq K \) where \( x, y \in I = [0,1] \). The function f has the Weak Cantor Intermediate Value Property (WCIVP) if there exists a Cantor set \( C \) between x and y such that \( f(C) \subseteq (f(x), f(y)) \).

**Theorem 1.** If \( f: I \to I \) has the CIVP, then f has the WCIVP.

**Proof.** Obvious.

**Example 1.** There exists a function \( f: I \to I \) that has the WCIVP but does not have the CIVP. Let \( S_y, y \in I \), be the collection of Cantor dense subsets of I constructed in [2]. Let \( r \in I \) be fixed. Let \( g: I \to \bigcup_{y \neq r} S_y \) where \( y \in I \) be 1-1 and onto. Define \( f(x) = g(y) \) where \( x \in S_r \) and \( y \neq r \). If \( x \in S_r \), let \( f(x) = 0 \). If \( x \) is not in any \( S_y \), let \( f(x) = 0 \).

Let \( a, b \in I \) and assume that \( f(a) < f(b) \). Let \( K \) be a Cantor set in \( (f(a), f(b)) \) such that \( K \subseteq S_y \) for some \( y \neq r \). Choose \( z \in K \) such that \( r \neq g^{-1}(z) = w \). Consider \( S_w \). If \( x \in S_w \), then \( f(x) = g(w) = z \) and \( f(S_w) \subseteq K \). By Cantor density there exists a Cantor set \( C \subseteq S_w \) such that \( C \subseteq (a, b) \) or \( C \subseteq (b, a) \). Therefore \( f(C) \subseteq (f(a), f(b)) \) and hence f has the WCIVP.
Let K be a Cantor set in \((f(a),f(b))\) such that \(K \subset S_r\). Since \(K\) contains no points of the range of \(f\), there exists no Cantor set \(C \subset I\) such that \(f(C) \subset K\). Therefore \(f\) does not have the CIVP.

**Theorem 2.** If \(f: I^2 \to I\) is a connectivity function, then \(f|I \times 0\) has the WCIVP. Moreover, the Cantor set can be selected such that \(f\) restricted to it is continuous.

**Proof.** It follows that a function \(I^2 \to I\) is a connectivity function if and only if it is peripherally continuous [3]. The function \(f: I^2 \to I\) is peripherally continuous if and only if \(U\) is an open subset of \(I^2\) containing a point \(x\) and \(V\) is an open subset of \(I\) containing \(f(x)\), then there is an open subset \(W\) of \(U\) containing \(x\) such that \(f(\text{bd}(W))\) is a subset of \(V\), where \(\text{bd}(W)\) is the boundary of \(W\).

Assume \(a, b \in I\) such that \(f(a) < f(b)\). Choose \(y \in I\) between \(a\) and \(b\) such that \(f(y) \in (f(a), f(b))\). Let \(\varepsilon = \min\{d(f(a), f(y)), d(f(y), f(b))\}\). Let \(U\) be a connected open subset of \(I^2\) with connected boundary \(C\) such that \(y \in U \subset \overline{U} \subset B(y, \eta/5)\) where \(\eta = \min\{d(y, a), d(y, b)\}\), and \(f(C) \subset B(f(y), \varepsilon/5)\). Then there exists \(y_0, y_1 \in I\) which are in \(C\) such that \(y_0 < y < y_1\).

\[
\begin{align*}
y_0 & \in B(y, \eta/5), & f(y_0) & \in B(f(y), \varepsilon/5), \\
y_1 & \in B(y, \eta/5), & f(y_1) & \in B(f(y), \varepsilon/5).
\end{align*}
\]

Clearly \(d(y_0, y) < \eta/5\) and \(d(y_1, y) < \eta/5\). Also \(d(f(y_0), f(y)) < \varepsilon/5\) and \(d(f(y_1), f(y)) < \varepsilon/5\).

Now there exist connected open subsets \(U_0\) and \(U_1\) of \(I^2\) with connected boundaries \(C_0\) and \(C_1\) such that
$y_0 \in U_0$, $\bar{U}_0 \subset B(y_0, \eta_0/5)$, $f(C_0) \subset B(f(y_0), \epsilon/5^2)$

and

$y_1 \in U_1$, $\bar{U}_1 \subset B(y_1, \eta_1/5)$, $f(C_1) \subset B(f(y_1), \epsilon/5^2)$

where $\eta_0 = d(y_0, y)$ and $\eta_1 = d(y_1, y)$. So $\eta_0 < \eta/5$ and $\eta_1 < \eta/5$.

Now $C_0$ has points $y_{00}, y_{01} \in I$ and $C_1$ has points $y_{10}, y_{11} \in I$ such that

$a < y_{00} < y_0 < y_{01} < y < y_{10} < y_1 < y_{11} < b,$

$y_{00}, y_{01} \in B(y_0, \eta_0/5), f(y_{00}), f(y_{01}) \in B(f(y_0), \epsilon/5^2)$,

$y_{10}, y_{11} \in B(y_1, \eta_1/5), f(y_{10}), f(y_{11}) \in B(f(y_1), \epsilon/5^2)$.

There exists connected open subsets $U_{00}, U_{01}, U_{10}, U_{11}$

of $I^2$ with connected boundaries $C_{00}, C_{01}, C_{10}, C_{11}$ such that

$y_{00} \in U_{00}$, $\bar{U}_{00} \subset B(y_{00}, \eta_{00}/5)$, $f(C_{00}) \subset B(f(y_{00}), \epsilon/5^3)$,

$y_{01} \in U_{01}$, $\bar{U}_{01} \subset B(y_{01}, \eta_{01}/5)$, $f(C_{01}) \subset B(f(y_{01}), \epsilon/5^3)$,

$y_{10} \in U_{10}$, $\bar{U}_{10} \subset B(y_{10}, \eta_{10}/5)$, $f(C_{10}) \subset B(f(y_{10}), \epsilon/5^3)$,

$y_{11} \in U_{11}$, $\bar{U}_{11} \subset B(y_{11}, \eta_{11}/5)$, $f(C_{11}) \subset B(f(y_{11}), \epsilon/5^3)$,

where $\eta_{00} = d(y_{00}, y_0), \eta_{01} = d(y_{01}, y_0), \eta_{10} = d(y_{10}, y_1)$, and $\eta_{11} = d(y_{11}, y_1)$.

Now $C_{00}$ has points $y_{000}, y_{001} \in I$, $C_{01}$ has points $y_{010}, y_{011} \in I$, $C_{10}$ has points $y_{100}, y_{101} \in I$, and $C_{11}$ has points $y_{110}, y_{111} \in I$ such that $a < y_{000} < y_{00} < y_{001} < y_0 < y_{01} < y_{011} < y < y_{10} < y_{100} < y_{101} < y_1 < y_{11} < y_{111} < b.$

$y_{000}, y_{001} \in B(y_{00}, \eta_{00}/5)$, $f(y_{000}), f(y_{001}) \in B(f(y_{00}), \epsilon/5^3)$,

$y_{010}, y_{011} \in B(y_{01}, \eta_{01}/5)$, $f(y_{010}), f(y_{011}) \in B(f(y_{01}), \epsilon/5^3)$,

$y_{100}, y_{101} \in B(y_{10}, \eta_{10}/5)$, $f(y_{100}), f(y_{101}) \in B(f(y_{10}), \epsilon/5^3)$,

and

$y_{110}, y_{111} \in B(y_{11}, \eta_{11}/5)$, $f(y_{110}), f(y_{111}) \in B(f(y_{11}), \epsilon/5^3)$.
Continuing this process let $a$ be a finite sequence of 0's and 1's of length $k$. Thus for $y_a$ we obtain

$$y_{a0} < y_a < y_{a1},$$
$$y_{a0} \in B(y_a, \eta_a/5),$$
$$y_{a1} \in B(y_a, \eta_a/5),$$
$$\eta_{a0} = d(y_{a0}, y_a),$$
$$\overline{a}_{a0} \subseteq B(y_{a0}, \eta_{a0}/5),$$
$$f(C_{a0}) \subseteq B(f(y_{a0}), \varepsilon/5^{k+2}),$$
$$\eta_{a1} = d(y_{a1}, y_a),$$
$$\overline{a}_{a1} \subseteq B(y_{a1}, \eta_{a1}/5),$$
$$f(C_{a1}) \subseteq B(f(y_{a1}), \varepsilon/5^{k+2})$$

where $\eta_{a0} = d(y_{a0}, y_a)$ and $\eta_{a1} = d(y_{a1}, y_a)$. Now $C_{a0}$ has points $y_{a00}, y_{a01} \in I$ and $C_{a1}$ has points $y_{a10}, y_{a11} \in I$ such that

$$y_{a00} < y_{a0} < y_{a01} < y_a < y_{a10} < y_{a1} < y_{a11},$$
$$y_{a00}, y_{a01} \in B(y_{a0}, \eta_{a0}/5), f(y_{a00}), f(y_{a01}) \in B(f(y_{a0}), \varepsilon/5^{k+2}),$$
$$y_{a10}, y_{a11} \in B(y_{a1}, \eta_{a1}/5), f(y_{a10}), f(y_{a11}) \in B(f(y_{a1}), \varepsilon/5^{k+2}).$$

We now claim that if $a$ and $\beta$ are finite binary sequences of equal length $n$ of the form $a = \gamma_0\mu$ and $\beta = \gamma_1\nu$ where $\gamma$ is of length $k \leq n-1$, then

1. $y_a < y_\beta$,
2. $3/4(\eta_0+\eta_1) \leq |y_a - y_\beta| \leq 5/4(\eta_0+\eta_1)$, and
3. $|f(y_a) - f(y_\beta)| < \varepsilon/2(5^k)$.

By construction $y_a < y_\beta$ and $y_{\gamma_0} < y_{\gamma_1}$. Thus $y_{\gamma_1} - y_{\gamma_0} = y_{\gamma_1} - y_\gamma + y_\gamma - y_{\gamma_0} = \eta_{\gamma_0} + \eta_{\gamma_1}$. Also

$$d(y_a, y_{\gamma_0}) < \eta_{\gamma_0}(1/5 + (1/5^2) + \cdots + (1/5^{n-k})) < \frac{1}{4} \eta_{\gamma_0}$$
and
$$d(y_\beta, y_{\gamma_1}) < \eta_{\gamma_1}(1/5 + (1/5^2) + \cdots + (1/5^{n-k})) < \frac{1}{4} \eta_{\gamma_1}.$$

From this it follows that (2) is true.
Now $|f(y_0) - f(y_\gamma)| < \varepsilon/5^{k+1}$ and $|f(y_{k+1}) - f(y_\gamma)| < \varepsilon/5^{k+1}$, 

$|f(y_\alpha) - f(y_\gamma)| < \varepsilon((1/5^k) + (1/5^{k+1}) + \cdots + (1/5^n))$, and 

$|f(y_\beta) - f(y_\gamma)| < \varepsilon((1/5^k) + (1/5^{k+1}) + \cdots + (1/5^n))$.

So $|f(y_\alpha) - f(y_\beta)| < 2\varepsilon((1/5^k) + (1/5^{k+1}) + \cdots + (1/5^n))$

$< 2\varepsilon(1/5^k)(1/4)$

$= \varepsilon/2(5^k)$.

Let $\alpha(n)$ denote a binary sequence with $n$ terms such that the first $n-1$ terms of $\alpha(n)$ is $\alpha(n-1)$. Define

$$Y_\alpha = \lim_{n \to \infty} Y_\alpha(n).$$

Then the previous claim holds true for infinite sequences $\alpha$ and $\beta$. We now prove that $f(y_\alpha) = \lim_{n \to \infty} f(y_\alpha(n))$. Each $C_\alpha(k+1)$ intersects $C_\alpha(k)$ since one point of $C_\alpha(k+1)$ is inside the interval formed by $C_\alpha(k)$ and one point is outside. Thus for any $\gamma$ of length $k$ the union of all sets $C_\gamma/\gamma$ is a connected set and its image points differ from $f(y_\gamma)$ by at most $(\varepsilon/5^k) + (\varepsilon/5^{k+1}) + \cdots = \varepsilon/4(5^{k-1})$. Since $f$ is a Darboux function (the image of connected sets is connected),

$$f(\overline{UC_\gamma/\gamma}) \subseteq \overline{f(UC_\gamma/\gamma)} \subseteq B(f(y_\gamma), \varepsilon/4(5^{k-1})).$$

Thus $d(f(y_\alpha), f(y_\alpha(n))) < \varepsilon/4(5^{n-1})$ where $\alpha(n) = \gamma$ and it follows that $f(y_\alpha(n))$ converges to $f(y_\alpha)$.

Now it follows that the function defined by the assignment $\alpha \to y_\alpha$ is a homeomorphism from a Cantor set to $S = \{y_\alpha\}$. Thus $S$ is a Cantor set and $f(S) \subseteq (f(a), f(b))$. So $f| I \times 0$ has the WCIVP and $f|S$ is continuous.

**Example 2.** The first example of [2] is an example of an almost continuous function $I \to I$ which does not have the WCIVP. For completeness that example will be described here. There exists a subset $G \subseteq I$ which intersects every
Cantor set in every interval \((a,b)\) but contains no Cantor set. Thus \(G \cap (a,b)\) contains \(c\) points. Let \(F_1 = \{(x,0) : x \notin G\}\). Consider the collection \(\{K\}\) of closed subsets of \(\mathbb{I}^2\) such that the \(x\)-projection of \(K\) has cardinality \(c\). The \(x\)-projection of any set in the collection is closed and contains a Cantor set. Hence it contains a point of \(G\).

Select a subset \(F_2 \subset \mathbb{I}^2\) by transfinite induction such that

1. \(F_2\) intersects each member of the collection \(\{K\}\) and
2. if \(p\) and \(q\) are distinct points of \(F_2\), then their \(x\)-projections are distinct points of \(G\).

Let \(F_3 = \{(t,1) : t \in I\) but \(t\) is neither in the \(x\)-projection of \(F_1\) nor in the \(x\)-projection of \(F_2\}\).

Let \(f = F_1 \cup F_2 \cup F_3\). Then the \(x\)-projection of \(f\) is \(I\) and \(f\) is the graph of a function \(f : I \rightarrow I\).

Remarks. The second example of \([2]\) is an example of a function \(I \rightarrow I\) which has the WCIIP but is not a Darboux function. Also, it follows that if \(f : I \rightarrow I\) is continuous then \(f\) has the P.

References


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