COMPACT AND REALCOMPACT
κ-METRIZABLE EXTENSIONS

by
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Shchepin [4,5] introduced the notions of $\kappa$-metrizability and capacity as a generalization of metric spaces and locally compact groups, and proved that the $\kappa$-metrizability is productive [6]. Bennett, Lewis and Luksic [1] showed that $\kappa$-metrizability is equivalent to faithful capacity and, that $\kappa$-metrizability is not closed-hereditary. In §2, we give a characterization of a $\kappa$-metric space with a compact $\kappa$-metrizable extension and, a characterization of when $\beta X$ is a $\kappa$-metrizable extension of a $\kappa$-metric space $X$. Next we give a characterization for a space $Y$ (especially, $\sigma X$) containing a $\kappa$-metric space $X$ as a dense subset to be $\kappa$-metrizable extension of $X$, and prove that if a subspace $X$ of the product $L$ of realcompact $\kappa$-metric spaces is either dense or open, then $\sigma X \subset L$.

In the following, we mean by a space a Tychonoff space and by a $\kappa$-metric space a normed $\kappa$-metric space (see 1.1 below) and assume familiarity with [2], whose terminology will be used throughout. We denote by $C(X)$ ($B(X)$) the set of (bounded) real-valued continuous functions defined on $X$, by $R$ the set of real numbers, by $RC(X)$ the set of regular closed subsets of $X$, by $U$ (or $V$) a free ultrafilter in $RC(X)$, by $\beta X(\sigma X)$ the Stone-Čech compactification (the real-compactification) of $X$, by $\beta f(\sigma f)$ the Stone-extension (the Hewitt-extension) of $f \in B(X)$ ($f \in C(X)$) and by $f_{\mathcal{F}}$ that
\{f_a; f_a \in C(X)\} is a decreasing sequence converging to \(\leftarrow\) pointwise.

I. Definitions and Preliminaries

Definition 1.1. A \(\kappa\)-metric \(d(x,C)\) on \(X\) is a mapping \(d: X \times RC(X) \to R\) satisfying the following \((K1) \sim (K4)\):

\((K1)\) \(d(x,C) = 0 \iff x \in C\).

\((K2)\) \(C \subseteq D \Rightarrow d(x,C) \geq d(x,D)\) for every \(x \in X\).

\((K3)\) \(d(x,C)\) is continuous in \(x\) for every \(C\).

\((K4)\) \(f \downarrow f\) for every increasing transfinite sequence \(\{C_a\}\) where \(f_a(x) = d(x,C_a), f(x) = d(x,D)\) and \(D = \text{cl}(\cup C_a)\).

Now we consider the following condition such that in \((K4)\) \(\{f_a\}\) converges to \(f\) uniformly, briefly

\((UK4)\) \(f_a \downarrow f\) (unif.).

A \(\kappa\)-metric \(d\) is said to be \(\text{normed}\) if \(d(x,\emptyset) < 1\) for every \(x \in X\). If \(d\) is a \(\kappa\)-metric, then \(d(x,C)/(1 + d(x,C))\) is always normed ([5], p. 79), and hence, in the following, we mean by a \(\kappa\)-metric space a normed \(\kappa\)-metric space, and by \(k(X)\) the set \(\{f_C(x); f_C(x) = d(x,C), C \in RC(X)\}\). \(C\) and \(D\) are \(f\)-\(\text{separated}\) [3] if \(\text{cl}_R^{-1}f(C) \cap \text{cl}_R^{-1}f(D) = \emptyset\) where \(C, D \in RC(X)\) and \(f \in C(X)\). \(D\) is said to be \(f_C\)-\(\text{separated}\) (\(f_C\)-\(\text{unseparated}\)) if \(\inf f_C(D) > 0\) (\(= 0\)) where \(f_C \in k(X)\) and \(D \in RC(X)\). The following lemma are well known or easily verified.

Lemma 1.2. (1) If \(L\) is a \(\kappa\)-metric space with a \(\kappa\)-metric \(d\) and \(X \subseteq L\), then \(X\) is \(\kappa\)-metrizable in each of the following cases [5]:

(i) \(X\) is dense (put \(d_x(x,C) = d(x,\text{cl}_L C)\)).

(ii) \(X\) is regular closed (put \(d_x(x,C) = d(x,C)\)).

(iii) \(X\) is open (by (i) and (ii)).
Let $X$ be dense in $L$, $f_a = g_a|X$ and $f = g|X$ where $g_a$, $g \in C(L)$. Then we have (i) $C \in RC(L) \Rightarrow X \cap C \in RC(X)$.

(ii) $D \in RC(X) \Rightarrow cl_L D \in RC(L)$. (iii) $f_a|f$ (unif.) iff $g_a|g$ (unif). (iv) If $L$ is compact, then $g_a|g = g_a|g$ (unif.).

Definition 1.3. Let $X$ be a $\kappa$-metric space. We consider the following (RC)- and (SRC)-conditions:

(RC) Let $C \cap D = \emptyset$, $C, D \in RC(X)$. If there exists $U \ni D$ such that every $E \in U$ is $f_C$-unseparated, then there exists $V \ni C$ such that $E$ and $F$ are $f_B$-unseparated for $E \in U$, $F \in V$ and $f_B \in k(X)$.

(SRC) $C \cap D = \emptyset$, $C, D \in RC(X) \Rightarrow D$ is $f_C$-separated.

The following is easily verified.

Lemma 1.4. (1) If $X$ is a $\kappa$-metric space, (SRC) $\Rightarrow$ (RC).

(2) If $X$ is pseudocompact, then $Z(f) \cap D = \emptyset$ implies $\inf|f(D)| > 0$ where $Z(f) \neq \emptyset$, $f \in C(X)$ and $RC(X) \ni D \neq \emptyset$.

(3) If $X$ is pseudocompact and $\kappa$-metrizable, then $X$ satisfies (SRC).

(4) Let $X$ be dense in $L$, $g \in B(L)$ and $f = g|X$. Then we have

(i) If $\inf|f(D)| > 0$ for every $D \in RC(X)$ with $Z(f) \cap D = \emptyset$, then $Z(g) = cl_L Z(f)$.

(ii) If $X$ is a $\kappa$-metric dense subspace satisfying (SRC) of $L$ and $f \in k(X)$, then $Z(g) = cl_L Z(f)$.

2. Compact and Realcompact $\kappa$-Metrizable Extensions

Definition 2.1. A space $L$ is said to be a $\kappa$-metrizable extension of a $\kappa$-metric space $X$ with a $\kappa$-metric $d$ if $X$
is dense in \( \mathcal{L} \) and there exists a \( \kappa \)-metric \( d_\mathcal{L} \) on \( \mathcal{L} \) such that 
\[
d_\mathcal{L}(x, D) = d(x, X \cap D) \quad \text{for } x \in \mathcal{X}, \ D \in \mathcal{R}_\mathcal{C}(\mathcal{L})
\]
(briefly \( d_\mathcal{L}|_{X = d} \)), and it is easily verified that \( d_\mathcal{L}(x, D)|_{X = d(x, C)} \)
for every \( x \in \mathcal{X} \) implies \( D = \operatorname{cl}_\mathcal{L}C \).

**Theorem 2.2.** Let \( \mathcal{X} \) be a \( \kappa \)-metric space. Then there exists a compact \( \kappa \)-metrizable extension \( \mathcal{L} \) of \( \mathcal{X} \) iff \( \mathcal{X} \) satisfies \((\text{UK4})\) and \((\text{RC})\).

**Proof.** \( \Rightarrow \). It suffices to show by 1.2(2) that \( \mathcal{X} \)
satisfies \((\text{RC})\). Let \( C \cap D = \emptyset, \ D \in \mathcal{U} \) and every \( E \in \mathcal{U} \) is
\( f_C \)-unseparated. \( \mathcal{L} \) being compact, there exists a point
\( p \in \mathcal{L} - \mathcal{X} \) with \( \mathcal{U} \ni p \). Now suppose \( f_\mathcal{X}(p) = 2h > 0 \) where
\( f_\mathcal{X}(z) = d_\mathcal{L}(z, \operatorname{cl}_\mathcal{L}C), \ z \in \mathcal{L}, \ d_\mathcal{L}|_{X = d} \). Then there exists
\( E \in \mathcal{U} \) such that \( p \in \operatorname{cl}_\mathcal{L}E \) and \( f_\mathcal{X}(E) > h \), which is impossible
because \( E \) is \( f_C \)-unseparated, and hence \( f_\mathcal{X}(p) = 0 \). Take
\( \mathcal{V} \) such that \( C \in \mathcal{V} \) and \( \mathcal{U} \ni p \). For \( f_B \in k(\mathcal{X}), \ E \in \mathcal{U}, \ F \in \mathcal{V}, \)
we have \( \operatorname{cl}_\mathcal{R}f_B(E) \cap \operatorname{cl}_\mathcal{R}f_B(F) \ni f_\mathcal{X}(p) \), which shows that \( \mathcal{X} \)
satisfies \((\text{RC})\).

\( \Leftarrow \). Let \( a(C) = \sup\{f_C(x); \ x \in \mathcal{X}\} \) and put \( I_C = [0, a(C)] \).
Then \( M = \mathcal{I}_C \) is a compact \( \kappa \)-metric space [6, Th. 2] and
\( \phi: \mathcal{X} \to M \) is a homeomorphism of \( \mathcal{X} \) to \( \phi(\mathcal{X}) = (f_C(x)), \ x \in \mathcal{X} \). We shall show that \( \mathcal{L} = \operatorname{cl}_M \phi(\mathcal{X}) \) is a \( \kappa \)-metrizable
extension of \( \mathcal{X} \). For \( y = (y_C) \in \mathcal{L} \), put \( f_\mathcal{X}(y) = y_C \). Obviously
\( f_\mathcal{X} \) is continuous. To show \( Z(f_\mathcal{X}) = \operatorname{cl}_\mathcal{L}C \), it suffices to
prove that \( f_\mathcal{X}(p) = 0 \) implies \( p \in \operatorname{cl}_\mathcal{L}C \). Suppose that
\( p \notin \operatorname{cl}_\mathcal{L}C \). Then there exists \( D \in \mathcal{R}_\mathcal{C}(\mathcal{X}) \) such that \( p \in \int_{\mathcal{L}} \operatorname{cl}_\mathcal{L}D \) and \( \operatorname{cl}_\mathcal{L}D \cap \operatorname{cl}_\mathcal{L}C = \emptyset \). Take \( \mathcal{U} \) such that \( D \in \mathcal{U} \) and
\( \mathcal{U} \ni p \). Since \( f_\mathcal{X}(p) = 0 \), it is easy to see that every \( E \in \mathcal{U} \)
is \( f_C \)-unseparated. By \((\text{RC})\), there exists \( \mathcal{V} \ni C \) such that
E and F are \( f_B \)-unseparated for \( E \in U \), \( F \in V \) and \( f_B \in k(X) \). Let \( V = q \). Let \( f_B \in k(X) \) and \( F \in V \). \( U \to p \) implies 
\[
\cap \{ cl_R f_B(E); E \in U \} = f^*_B(p) = p_B. 
\]
On the other hand, 
\[
A(F) = \{ cl_R f_B(E) \cap cl_R f_B(F); E \in U \} 
\] 
is a collection of non-empty closed sets in \([0,1]\) with the finite intersection property. Thus \( \cap A(F) = f^*_B(p) \). Since \( F \) is an arbitrary element of \( U \), we have 
\[
\cap \{ A(F); F \in V \} = f^*_B(p). 
\]
Since 
\[
\cap \{ cl_R f_B(E); E \in V \} = f_B^*(q), \quad \text{we have} \quad f_B^*(p) = f_B^*(q), \quad \text{i.e.,} 
\]
\( p_B = q_B \) for every \( f_B \in k(X) \), and hence \( p = q \), a contradiction. Thus we have \( Z(f^*_C) = cl_L C \) for \( C \in RC(X) \). From this fact and (UK4), it is easily verified that the function \( d_L \) defined by \( d_L(z,D) = f^*_C(z) \) with \( cl_L C = D \) is \( \kappa \)-metric on \( L \) and \( d_L|X = d \), and hence \( L \) is a \( \kappa \)-metrizable extension of \( X \).

**Lemma 2.3.** Let \( X \) be dense in a \( \kappa \)-metric space \( L \), and \( Y \) a space containing \( X \) as a dense subset such that every \( f_C \in k(X) \) has the continuous extension \( f^*_C \) on \( Y \) with \[
\text{cl}_Y C = Z(f^*_C) \quad \text{where} \quad f_C(x) = d_L(x, cl_L C) \quad \text{(see 1.2(1)(i))}.
\]
Then a continuous mapping \( \phi \) from \( Y \) onto \( L \) that leaves \( X \) pointwise fixed is a homeomorphism, so \( Y \) is a \( \kappa \)-metrizable extension of \( X \).

**Proof.** Let \( p, q \in Y - X \), \( p \neq q \) and \( \phi(p) = \phi(q) = r \in L - X \). Then there exist \( C, D \in RC(X) \) such that 
\[
\text{cl}_Y C \cap \text{cl}_Y D = \emptyset, \quad p \in \text{int}_Y \text{cl}_Y C, \quad q \in \text{int}_Y \text{cl}_Y D, \quad f_C^*(p) = 0 \quad \text{and} \quad f_C^*(q) = 2h > 0. 
\]
Thus we have \( r \in \phi(\text{cl}_Y C) \subset \text{cl}_L \phi(C) = \text{cl}_L C \) and \( d_L(r, \text{cl}_L C) = 0. \) Similarly, \( r \in \text{cl}_L D \) and we may assume \( d_L(r, \text{cl}_L D) > h \), a contradiction, i.e., \( \phi \) is \( 1-1 \).
By the same method, we obtain the closedness of \( \phi \), so \( \phi \) is homeomorphic.
As a space $Y$ in Lemma 2.3, we can take the following:

$Y = X \cup (\beta X - \bigcup \{Z(\beta f_C) = \text{cl}_{\beta X} C; C \in \text{RC}(X)\}). \ Z(\phi f) = \text{cl}_{\text{ux}} Z(f)$ for $f \in C(X)$ \cite{2} implies $\nu X \subset Y$.

**Theorem 2.4.** If $X$ is a $\kappa$-metric space, then we have

1. $\beta X$ is a $\kappa$-metrizable extension of $X$ iff $X$ satisfies (SRC) and (UK4).

2. If $X$ is pseudocompact, then $\beta X$ is a $\kappa$-metrizable extension of $X$ iff $X$ satisfies (UK4).

**Proof.** (1) $\Rightarrow$ If $Z(f) \cap Z(g) = \emptyset, f, g \in B(X)$, then $\text{cl}_{\beta X} Z(f) \cap \text{cl}_{\beta X} Z(g) = \emptyset$ \cite{2}, and hence $X$ satisfies (SRC) because $\beta X$ is a $\kappa$-metrizable extension of $X$. On the other hand, $X$ satisfies (UK4) by 2.2.

$\Leftarrow$ Since there exists a compact $\kappa$-metrizable extension $L$ of $X$ by 2.2, and $X$ satisfies (SRC), it is easily verified that we can take $\beta X$ as a space $Y$ in 2.3. On the other hand, there exists a continuous mapping from $\beta X$ onto $L$ that leaves $X$ pointwise fixed, and hence $\beta X$ is a $\kappa$-metrizable extension of $X$ by 2.3.

(2) From (1) and 1.4(3).

**Theorem 2.5.** If a subspace $X$ of the product $L$ of realcompact $\kappa$-metric spaces $\{X_a; a \in A\}$ is either dense or open, then we have

1. $\nu X$ is a $\kappa$-metrizable extension of $X$ and $\nu X \subset L$.

2. If $X$ is pseudocompact, then $X$ is $C^*$-embedded in $L$ and $\beta X \subset L$, especially, if $X$ is dense, $\beta X = L$.

**Proof.** (1) (i) $X$ is dense. Since $L$ is $\kappa$-metrizable \cite{6} and realcompact \cite{2}, $X$ is a $\kappa$-metric space by 1.2(1),
and hence there is a mapping $\phi$ of $uX$ to $L$ that leaves $X$ pointwise fixed [2]. $X \subseteq \phi(uX) \subseteq L$, so $\phi(uX)$ is $\kappa$-metrizable by 1.2(1), and hence $uX$ is homeomorphic to $\phi(uX)$ by 2.3.

(ii) $X$ is open. Apply a similar argument above. (2) follows from the fact that the pseudocompactness of $X$ implies $\beta X = uX$.

Lemma 2.6. Let $X$ be dense in $L$, $g_a, g \in C(L)$ and $f_a|f$ where $f_a = g_a|X$ and $f = g|X$. Then $g_a|f$ if and only if $X$ satisfies the following:

(L-K4) for each $p \in L - X$ and each $f_a \in C(X)$ with $f_a|f \in C(X)$, there exists $U \in \mathcal{R}(X)$ such that $p \in \text{int}_L cl_L U$ and $f_a|f$ (unif. on $U$).

Proof. Since the implication $\Rightarrow$ follows from 1.2, we shall show $\Leftarrow$. Let $p \in L - X$ and take $V \in \mathcal{R}(X)$ with $p \in \text{int}_L cl_L V$. Let us put $F_a = \{x \in V; f_a(x) - f(x) > \varepsilon\}$ for every $a$. Obviously $a > b \Rightarrow F_a \supseteq F_b$.

Case 1). $F_a = \emptyset$ for some $a$. Put $U = V$. Thus we have $F_a \neq \emptyset$ for every $a$.

Case 2). $p \notin cl_L F_a$ for some $a$. Then there exists $W \in \mathcal{R}(X)$ such that $p \in \text{int}_L cl_L W$ and $cl_L W \cap cl_L F_a = \emptyset$. Put $U = V \cap W$.

Case 3). $p \notin cl_L F_a$ for every $a$. Since $g_a(x) - g(x) = f_a(x) - f(x) > \varepsilon$ for every $x \in F_a$, and $p \in cl_L F_a$, we have $g_a(p) - g(p) > \varepsilon$, a contradiction.

Using 2.6, the following two theorems are easily verified.
Theorem 2.7. A space $L$ is a $\kappa$-metrizable extension of a $\kappa$-metric space $X$ iff $X$ satisfies (L-K4) and every $f_C \in k(X)$ has the continuous extension $g$ over $L$ with $Z(g) = cL_C$.

Theorem 2.8. Let $X$ be a $\kappa$-metrizable space. Then the following are equivalent:

1. $X$ satisfies (UX-K4).
2. $\text{UX}$ is a $\kappa$-metrizable extension of $X$.
3. There exists a realcompact $\kappa$-metrizable extension of $X$.

Quite recently, we proved the following: (1) If $\beta X$ is $\kappa$-metrizable, then $X$ is pseudocompact. (2) If $X$ is locally compact and $\beta X - X$ is $\kappa$-metrizable, then $X$ is pseudocompact. (3) If $X$ is a pseudocompact $\kappa$-metric space, and $Y$ is a compact $\kappa$-metrizable extension of $X$, then $\beta X = Y$.

References


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