DECOMPOSITIONS FOR CLOSED MAPS

by

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Introduction

Well-known decomposition theorems for closed maps are given by the following type (L):

(L) For spaces $X$, $Y$ and a closed map $f: X \to Y$,

$$Y = \bigcup_{n=1}^{\infty} Y_n,$$

where $f^{-1}(y)$ is compact for each $y \in Y_0$ and $Y_n$ is closed discrete in $Y$ for each $n \geq 1$.

First, Morita [10] showed that the following special type (M) of (L) holds for any paracompact and locally compact space $X$.

(M) For spaces $X$, $Y$ and a closed map $f: X \to Y$,

$$Y = \bigcup_{n=1}^{\infty} Y_n,$$

where $f^{-1}(y)$ is compact for each $y \in Y_0$ and $Y_n$ is closed discrete in $Y$.

Lašnev [7] proved that (L) holds for any metric space $X$. Subsequently, Filippov [4] extended this result, showing that (L) holds for any paracompact $M$-space $X$. Moreover, many mathematicians extended Lašnev's "decomposition theorem" for several generalized metric spaces (for example, see [14] and [15] etc.). In particular, recently, Chaber [2] proved that (L) holds for any regular $\sigma$-space $X$.

Now, let us consider the following modifications (wL) and (wL)' of (L), which are the weakening of the compactness of $f^{-1}(y)$ in (L):

(wL) In (L), the $f^{-1}(y)$ is Lindelöf for each $y \in Y_0$.

(wL)' In (L), the $f^{-1}(y)$ is $\omega_1$-compact for each $y \in Y_0$. 
Here, we say that a space $X$ is $\omega_1$-compact if every uncountable subset of $X$ has a cluster point in $X$. Every Lindelöf space is $\omega_1$-compact. It should be remarked that the condition (wL) was considered in [2], where it was labeled (**). The modifications (wM) and (wM)' of (M) are to be made similarly. That is,

(wM) In (M), the $f^{-1}(y)$ is Lindelöf for each $y \in Y_0$.
(wM)' In (M), the $f^{-1}(y)$ is $\omega_1$-compact for each $y \in Y_0$.

In Section 1, we investigate the decomposition types (wL), (wL)', and some variations of these types. Nagami [12] introduced the notion of $\Sigma$-spaces as a generalization of $\sigma$-spaces and M-spaces. We prove that (wL)' holds for any $\Sigma$-space $X$, thus (wL) holds for any strong $\Sigma$-space $X$. We shall remark that it is impossible to replace (wL) with (L) for every regular Lindelöf $\Sigma$-space $X$.

In Section 2, we discuss the decomposition types (M), (wM) and (wM)'. We prove that (M) holds for any space $X$ dominated by compact sets $X_\alpha'$, or determined by a point-countable cover of compact sets $X_\alpha$. If the $X_\alpha$'s are Lindelöf; $\omega_1$-compact, then (wM); (wM)' holds respectively.

We assume that all spaces are $T_1$, and all maps are continuous and onto.

1. Decomposition Types (wL) and (wL)'

Let $A = \{A_\lambda: \lambda \in \Lambda\}$ be a collection of subsets of a space $X$. We say that $A$ is hereditarily closure-preserving (abbreviated by HCP) if any collection $\{B_\lambda: \lambda \in \Lambda\}$ with $B_\lambda \subset A_\lambda$ for each $\lambda \in \Lambda$ is closure-preserving (that is, $\cup\{B_\lambda: \lambda \in \Lambda'\} = \cup\{B_\lambda: \lambda \in \Lambda'\}$ for any $\Lambda' \subset \Lambda$).
Lemma 1.1 ([15, Lemma 5.4]). Let $Y$ be a space, and $J$ a HCP collection of closed sets in $Y$. For each $n \geq 1$, let

$$Y_n = \bigcup \{ F_1 \cap \cdots \cap F_n : F_i \in J \text{ and } F_1 \cap \cdots \cap F_n \text{ is a non-empty finite set} \}.$$ 

Then each $Y_n$ is a closed discrete subset of $Y$.

Proof. It is routinely verified that each collection

$$\{ F_1 \cap \cdots \cap F_n : F_i \in J \text{ (i = 1, \cdots, n)} \}$$

is HCP. Thus each subset of $Y_n$ is closed in $Y$. Hence each $Y_n$ is closed discrete in $Y$.

Let $K$ be a cover of a space $X$. A cover $J$ is called a (mod $K$)-net for $X$ [9], if, whenever $K \subset U$ with $K \in K$ and $U$ open in $X$, there is some $F \in J$ such that $K \subset F \subset U$.

Lemma 1.2. Let $X$ be a space, and $K$ a cover of $X$ by countably compact sets. If $X$ has a (mod $K$)-net $J$ which is countable, then it is $\omega_1$-compact.

Proof. Assume the contrary. Then there is an uncountable closed discrete subset $D$ of $X$. For each $K \in K$, since $K$ is countably compact, $D \cap K$ is at most finite. Thus there is some $F_K \in J$ such that $K \subset F_K$ and $D \cap F_K$ is at most finite. But, since $J$ is countable and $K$ is a cover of $X$, we may assume that $\{ F_K : K \in K \}$ is a countable cover of $X$. Thus there is some $F_K \in J$ such that $D \cap F_K$ is infinite. This is a contradiction.

A space $X$ is called a $\Sigma$-space [12] ($\Sigma^*$-space [13]) if it has a $\sigma$-locally finite ($\sigma$-HCP) closed (mod $K$)-net $J = \bigcup_{n=1}^{\infty} J_n$ for some closed cover $K$ by countably compact sets. Here we can assume that $J_n$ is a locally finite (HCP) closed
cover of $X$ and $J_n \subseteq J_{n+1}$ for each $n \geq 1$. It should be noted that any sequence $\{x_n\}$ such that

$$x_n \in \cap \{F \in J_n : x \in F\}$$

for some $x \in X$ has a cluster point in $X$. Such a sequence $\{J_n\}$ is called a $\Sigma$-net ($\Sigma^*$-net) of $X$.

**Theorem 1.3.** If $X$ is a $\Sigma$-space, then (wL)' holds.

**Proof.** Let $\{J_n\}$ be a $\Sigma$-net of $X$. We may assume that each $J_n$ is finitely multiplicative. For each $n \geq 1$, put

$$Y_n = \bigcup \{f(F) \cap f(F') : F, F' \in J_n \text{ and } f(F) \cap f(F') \text{ is a non-empty finite set}\}.$$

Since each $J_n$ is locally finite in $X$ and $f$ is a closed map, $\{f(F) : F \in J_n\}$ is HCP. It follows from Lemma 1.1 that $Y_n$ is closed discrete in $Y$ for each $n \geq 1$. Put $Y_0 = Y \setminus \bigcup_{n=1}^{\infty} Y_n$. Pick any $y \in Y_0$. Let us show that $f^{-1}(y)$ is $\omega_1$-compact. It suffices to show from Lemma 1.2 that the subcollection $\{F \in J_n : F \cap f^{-1}(y) \neq \emptyset\}$ of $J_n$ is finite for each $n \geq 1$.

Assume the contrary. Then there are some $m \geq 1$ and a sequence $\{F_n\}$ of distinct members of $J_m$ such that each $F_n$ meets $f^{-1}(y)$. Pick an $x_0 \in f^{-1}(y)$. By the choice of $\{J_n\}$, $E_n = \cap \{F \in J_n : x_0 \in F\}$ and $F_n$ belong to $J_n$ for each $n \geq m$. Since $y \in f(E_n) \cap f(F_n \setminus Y_n)$, $f(E_n) \cap f(F_n)$ is an infinite set. So we can choose a sequence $\{y_n\}_{n \geq m}$ of distinct points in $Y$ such that $y_n \in f(E_n) \cap f(F_n)$. For each $n \geq m$, pick two points $p_n$ and $q_n$ from $E_n \cap f^{-1}(y_n)$ and $F_n \cap f^{-1}(y_n)$, respectively. Since $\{p_n\}_{n \geq m}$ has a cluster point in $X$, $\{y_n\}_{n \geq m}$ has also a cluster point in $Y$. On the other hand, $\{q_n : n \geq m\}$ is closed discrete in $X$, because $\{F_n : n \geq m\}$ is locally finite in $X$. Thus, since $f$ is a closed map, $\{y_n : n \geq m\}$ is also closed.
discrete in $Y$. This contradiction completes the proof.

A space is called a strong $\Sigma$-spaces \cite{12} if it satisfies the definition of a $\Sigma$-space for some closed cover $C$ by compact sets instead of $K$. By \cite[Proposition 4.4]{6}, an $\omega_1$-compact, strong $\Sigma$-space is Lindelöf. So Theorem 1.3 yields

**Corollary 1.4.** If $X$ is a strong $\Sigma$-space, then (wL) holds.

**Remark.** In the previous corollary, by \cite[Example 5.12]{15} or \cite[Example 1.2]{2}, we cannot replace "Lindelöf" with "compact" even if $X$ is regular $\sigma$-compact. Chaber \cite{2} showed that Corollary 1.4 holds under the assumption of $X$ being a $k$-space.

Next, we proceed with some variations of (wL) or (wL)'.

**Theorem 1.5.** Let $f: X \to Y$ be a closed map. If $X$ is a $\Sigma^*$-space, then $Y = Y_0 \cup \left( \bigcup_{n=1}^{\infty} Y_n \right)$, where $f^{-1}(y)$ is $\omega_1$-compact for each $y \in Y_0$ and $Y_n$ is a discrete set such that $\bigcup_{i=1}^{n} Y_i$ is closed in $Y$ for each $n \geq 1$.

**Proof.** Let $\{J_n\}$ be a $\Sigma^*$-net of $X$. For each $n \geq 1$, put

$$C_n = \{f(nC) \cap f(F) : C \subseteq J_n, F \subseteq J_n \text{ and } f(nC) \cap f(F) \text{ is a non-empty finite set} \}.$$

Since each $J_n$ is a closure-preserving closed cover of $X$ and $f$ is a closed map, each $C_n$ is a closure-preserving collection by finite sets. Then $Y'_n = \bigcup C_n$ is a closed set in $Y$ with a closure-preserving cover by finite sets for each $n \geq 1$. By \cite[Theorem 1]{16}, we have $Y'_n = \bigcup_{k=1}^{\infty} Y'_{nk}$, where $Y'_{nk}$ is a discrete
subset and \( \bigcup_{i=1}^{k} Y_{n_{i}} \) is closed in \( Y_n \) for each \( k > 1 \). Hence \( \bigcup_{n=1}^{\infty} Y_{n} \) can be represented as \( \bigcup_{n=1}^{\infty} Y_{n} \) described in the theorem. Put \( Y_{0} = Y \setminus \bigcup_{n=1}^{\infty} Y_{n} \). Pick any \( y \in Y_{0} \). By a similar way as in the proof of Theorem 1, we can show that the subcollection \( \{ F \in J_{n} : F \cap f^{-1}(y) \neq \emptyset \} \) of \( J_{n} \) is finite for each \( n > 1 \). Thus, it follows from Lemma 1.2 that \( f^{-1}(y) \) is \( \omega_{1} \)-compact.

The proof is complete.

Let \( \{ F_{n} \} \) be a sequence of subsets of a space \( X \). We say that \( \{ F_{n} \} \) converges to \( E \subset X \) if for any open set \( V \) with \( E \subset V \) there is some \( m \geq 1 \) such that \( V \) contains \( F_{n} \) for each \( n \geq m \). The following lemma is a modification of Lašnev's lemma in [7].

Lemma 1.6. Let \( \{ F_{n} \} \) be a sequence of non-empty closed sets in a regular space \( X \), and let \( E \) be a closed Lindelöf subspace of \( X \). If \( \{ F_{n} \} \) converges to \( E \) and each \( F_{n} \) is disjoint from \( E \), then \( K = E \cap \bigcup_{n=1}^{\infty} F_{n} \) is compact.

Proof. Suppose that \( K \) is not countably compact. Since \( X \) is regular and \( K \) is Lindelöf, there is an increasing sequence \( \{ U_{k} \} \) of open sets in \( X \) such that \( K \subset \bigcup_{k=1}^{\infty} U_{k} \) and \( (U_{k} \setminus U_{k-1}) \cap K \neq \emptyset \) for each \( k > 1 \), where \( U_{0} = \emptyset \) (This choice is seen in the proof of [2, Theorem 1.1]). Since each \( U_{k} \setminus U_{k-1} \) meets infinitely many \( F_{n} \)'s, there are two sequences \( \{ n_{k} \} \) and \( \{ x_{k} \} \) such that \( n_{k} < n_{k+1} \) and \( x_{k} \in (U_{k} \setminus U_{k-1}) \cap F_{n_{k}} \) for each \( k > 1 \). Put \( G = X \setminus \{ x_{k} : k \geq 1 \} \). Then \( G \) is an open set in \( X \). If \( x \in E \setminus K \), then \( x \notin \bigcup_{n=1}^{\infty} F_{n} \supset \bigcup_{k=1}^{\infty} F_{n_{k}} \supset \{ x_{k} : k \geq 1 \} = X \setminus G \).

Let \( x \in E \cap K \). Take some \( m \geq 1 \) with \( x \in U_{m} \). Since
$x_k \not\in \bigcup_{k=1}^{m} U_m$ for each $k > m$, $x \not\in \{x_k : k > m\}$. Since $x \notin E$ and $x_k \notin F_n$, $x \neq x_k$ for each $k > 1$. These imply $x \not\in \{x_k : k > 1\} = X \setminus G$. Thus, we have $E \subset G$. But $G$ does not contain any $F_n$'s. This contradicts to the fact that $\{F_n\}$ converges to $E$. So we conclude that $K$ is countably compact. Therefore, $K$ is compact.

Using Lemma 1.6 instead of [14, Lemma 2.1], the proof of the following theorem is quite parallel to that of [14, Theorem 1.2]. So the details are left to the reader.

Theorem 1.7. Let $f : X \to Y$ be a closed map. If $X$ is a regular semi-stratifiable space [3], then

$$\{y \in Y : f^{-1}(y) \text{ is Lindelöf}\} = Y_0 \cup (\bigcup_{n=1}^{\infty} Z_n),$$

where $f^{-1}(y)$ is compact for each $y \in Y_0$ and $Z_n$ is closed discrete in $Y$ for each $n > 1$.

Since $\sigma$-spaces are semi-stratifiable, strong $E$-spaces, the following is an immediate consequence of Corollary 1.4 and Theorem 1.7.

Corollary 1.8 ([2, Theorem 1.1]). If $X$ is a regular $\sigma$-space, then (L) holds.

2. Decomposition Types (M), (wM) and (wM)'

Let us recall basic definitions concerning weak topologies. Let $X$ be a space, and $C$ be a cover of $X$. We say that $X$ is determined by $C$ [5], or $X$ has the weak topology with respect to $C$, if $A \subset X$ is closed (open) in $X$ whenever $A \cap C$ is relatively closed (relatively open) in $C$ for each $C \in C$. Every space is determined by any open cover of it.
Let $C$ be a closed cover of $X$. We say that $X$ is dominated by $C$ if for any subcollection $C'$ of $C$ the union $\bigcup C'$ is closed in $X$ and is determined by $C'$. Every space is dominated by any HCP closed cover of it. We remark that if $X$ is dominated by $C$, then it is determined by $C$, but the converse is not true.

The following elementary facts will be often used later on. The proof is straightforward, so we omit it.

**Lemma 2.1.** (1) Let $f: X \to Y$ be a quotient map. If $X$ is determined by a cover $C$, then $Y$ is determined by a cover $f(C) = \{f(C): C \in C\}$.

(2) Let $X$ be determined by a cover $\{X_\alpha\}$. If $X_\alpha \subset X'_\alpha$ for each $\alpha$, then $X$ is determined by $\{X'_\alpha\}$.

Recall that a collection $C$ of subsets of $X$ is point-countable if each $x \in X$ is in at most countably many $C \in C$.

**Theorem 2.2.** (1) If a space $X$ is dominated by a cover $C$ of compact (Lindelöf; $\omega_1$-compact) sets, then $(M)$ $((\text{WM}); (\text{WM})')$ holds.

(2) If a space $X$ is determined by a point-countable cover $C$ of compact (Lindelöf; $\omega_1$-compact) sets, then $(M)$ $((\text{WM}); (\text{WM})')$ holds.

**Proof.** (1): Let $C = \{X_\alpha\}$ with the index set well-ordered. Suppose that each $X_\alpha$ is $\omega_1$-compact (for the other cases, the proofs are similar). Let $L_\alpha = X_\alpha \setminus \bigcup_{\beta < \alpha} X_\beta$ for each $\alpha$. Since $\{L_\alpha\}$ is a cover of $X$ and $L_\alpha \subset X'_\alpha$, it suffices to show that
$Y_1 = \{y \in Y : f^{-1}(y) \text{ meets uncountably many } L_\alpha \text{'s}\}$ is closed discrete in $Y$.

Claim 1. For any $D \subseteq Y_1$ with cardinality $\leq \omega_1$, $D$ is closed in $Y$:

Let $D = \{y_\beta : \beta < \kappa\}$, where $\kappa \leq \omega_1$. For each $\beta < \kappa$, we can choose some $x_\beta$ and $a(\beta)$ such that $x_\beta \in f^{-1}(y_\beta) \cap L_\alpha(\beta)$ and $a(\beta) \neq a(\beta')$ for $\beta \neq \beta'$. Let $E = \{x_\beta : \beta < \kappa\}$. We show that $E$ is closed in $X$. First, $E \cap X_0$ is at most one point, hence closed in $X_0$. Assume that $E \cap X_\lambda$ is closed in $X_\lambda$ for each $\lambda < \eta$. Let $E_\eta = (E \cap (\cup_{\lambda < \eta} X_\lambda)) \cap X_\eta$. Then $E_\eta \subseteq \cup_{\lambda < \eta} X_\lambda$, and $E_\eta \cap X_\lambda = (E \cap X_\lambda) \cap X_\eta$ is closed in $X_\lambda$ for each $\lambda < \eta$. Thus $E_\eta$ is closed in $X$. But, $E \cap X_\eta = (E \cap L_\eta) \cup E_\eta$, and $E \cap L_\eta$ is at most one point. Hence $E \cap X_\eta$ is closed in $X_\eta$. Thus $E$ is closed in $X$. Since $f$ is a closed map and $D = f(E)$, $D$ is closed in $Y$.

Claim 2. For any $Y' \subseteq Y_1$ and for any $\omega_1$-compact set $K \subseteq Y$, $Y' \cap K$ is closed in $K$:

If $Y' \cap K$ is countable, by Claim 1, $Y' \cap K$ is closed in $Y$. Hence it is closed in $K$. If $Y' \cap K$ is not countable, then there is a subset $D$ of $Y' \cap K$ such that the cardinality of $D$ is $\omega_1$ and $D$ has a cluster point in $K \setminus D$. But, by Claim 1, $D$ is closed in $Y$. This contradiction implies that $Y' \cap K$ is countable, hence closed in $K$.

Now, $X$ is determined by a cover $\mathcal{C}$ of $\omega_1$-compact sets. Since $f$ is quotient, by Lemma 2.1 (1), $Y$ is determined by a cover $f(\mathcal{C})$ of $\omega_1$-compact sets. Thus it follows from Claim 2 that $Y_1$ is closed discrete in $Y$. Hence $(\text{wM})'$ holds.
(2): Let $C = \{X_\alpha\}$. Suppose that each $X_\alpha$ is $\omega_1$-compact (for the case of $X_\alpha$ being Lindelöf, the proof is similar). Let

$$Y_1 = \{y \in Y: \text{no countable } C' \subset C \text{ covers } f^{-1}(y)\}.$$

Let $D = \{y_\beta: \beta < \kappa\}$, where $\kappa \leq \omega_1$, be a subset of $Y_1$. Then there is some

$$x_\beta \in f^{-1}(y_\beta) \cup \{x_\gamma: x_\gamma \in X_\alpha \text{ for some } \gamma < \beta\}$$

for each $\beta < \kappa$. Let $E = \{x_\beta: \beta < \kappa\}$. Since each $E \cap X_\alpha$ is at most one point, $E$ is closed in $X$. Then $D = f(E)$ is closed in $Y$. Thus Claim 1 in (1) is also valid.

Next, suppose that each $X_\alpha$ is compact. Let

$$Y_1^* = \{y \in Y: \text{no finite } C' \subset C \text{ covers } f^{-1}(y)\}.$$

Then we can show that any countable $D = \{y_n: n \geq 1\} \subset Y_1^*$ is closed in $Y$. Indeed, there is some

$$x_n \in f^{-1}(y_n) \cup \{x_{ij}: i, j \leq n\} \text{ for each } n \geq 1,$$

where $\{x_{ij}: j \geq 1\} = \{x_\alpha: x_i \in X_\alpha\} \text{ for each } i \geq 1$. Let $E = \{x_n: n \geq 1\}$. Since each $E \cap X_\alpha$ is at most finite, $E$ is closed in $X$. Then $D = f(E)$ is closed in $Y$. Thus the modification of Claim 1 in (1) where "$\omega_1$" is replaced with "$\omega_0$" is valid.

As in the proof of (1), if the $X_\alpha$'s are respectively $\omega_1$-compact; compact, we can show that the set $Y_1; Y_1^*$ is closed discrete in $Y$, hence $(wM)'$; $(M)$ holds. The proof of Theorem 2.2 is complete.

As is well-known, every CW-complex is dominated by a cover of compact (metric) sets. So, by Theorem 2.2 (1), we have
Corollary 2.3. If X is a CW-complex (more generally, a chunk-complex in the sense of [1]), then (M) holds.

A space is called a $k_\omega$-space [8] (Morita [11] calls it a space of the class $\mathcal{G}'$) if it is determined by a countable cover of compact sets. Such a space is characterized as a quotient image of a locally compact Lindelöf space [11].

By Theorem 2.2 (2), if X is a $k_\omega$-space, then (M) holds. It should be noted by Remark to Corollary 1.4 that we cannot replace "$k_\omega$-space" with "$\sigma$-compact space."

Let us call a space locally $k_\omega$ (locally Lindelöf) if each point has a neighborhood which is $k_\omega$ (Lindelöf), where the neighborhood is not necessarily open. Every locally compact space is locally $k_\omega$, and every locally $k_\omega$-space is locally Lindelöf. Recall that a space X is meta-Lindelöf if every open cover of X has a point-countable open refinement.

Morita [10] showed that if X is a paracompact and locally compact space, then (M) holds. We can extend this result as follows.

Proposition 2.4. If X is a meta-Lindelöf and locally $k_\omega$-space, then (M) holds.

Proof. By the assumptions of X, X is determined by a point-countable open cover $\{V_\alpha\}$, where each $V_\alpha$ is contained in a space determined by a countable cover $\{K_{an} : n \geq 1\}$ of compact sets. Let $C = \{V_\alpha \cap K_{an}\}$. Then it is routinely verified that X is determined by the point-countable cover $C$. But, by Lemma 2.1 (1), Y is determined by the cover...
f(\mathcal{C}). Since f(V \cap K_{an}) \subseteq f(K_{an}) for each \alpha and n, by Lemma 2.1 (2), Y is determined by the cover \{f(K_{an})\} of compact sets. Let

$$Y^*_1 = \{y \in Y: \text{no finite } \mathcal{C}' \subseteq \mathcal{C} \text{ covers } f^{-1}(y)\}.$$ 

By the proof of Theorem 2.2 (2) for the compact case, we can show that \(Y^*_1\) is closed discrete in \(Y\). Hence (M) holds.

**Proposition 2.5.** Let \(X\) be a locally Lindelöf space. If \(X\) is subparacompact (meta-Lindelöf), then \((wL) ((wM))\) holds.

**Proof.** If \(X\) is subparacompact, then \(X\) has a \(\sigma\)-locally finite closed cover \(J = \bigcup_{n=1}^{\infty} J_n\) of Lindelöf sets, where each \(J_n\) is locally finite in \(X\). Let \(X_n = \bigcup_j J_n\) and \(Y_n = f(X_n)\) for each \(n \geq 1\). Then each \(X_n\) is dominated by \(J_n\) and \(Y_n = f|_{X_n}\) is closed. Thus, by Theorem 2.2 (1), \(Y_n = Y_{n0} \cup Y_{n1}\), where \(g^{-1}_n(y)\) is Lindelöf for each \(y \in Y_{n0}\) and \(Y_{n1}\) is closed discrete in \(Y_n\), hence in \(Y\). Let \(Y_0 = Y \setminus \bigcup_{n=0}^{\infty} Y_{n1}\) and \(Y_n = Y_{n1}\) for each \(n \geq 1\). Then the sets \(Y_n\), \(n \geq 0\), satisfy the desired property in \((wL)\).

If \(X\) is meta-Lindelöf, then \(X\) is determined by a point-countable open cover \(\{V_{\alpha}\}\) such that each \(V_{\alpha}\) is contained in a Lindelöf set \(L_{\alpha}\). But, by Lemma 2.1 (1), \(Y\) is determined by a cover \(\{f(V_{\alpha})\}\). Since \(f(V_{\alpha}) \subseteq f(L_{\alpha})\) for each \(\alpha\), by Lemma 2.1 (2), \(Y\) is determined by a cover \(\{f(L_{\alpha})\}\) of Lindelöf sets. Thus, in view of the proof of Theorem 2.2 (2), the assertion for the parenthetic part holds.

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