**TOPOLOGY PROCEEDINGS**

**Volume 11, 1986**

Pages 1–5

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by

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**Topology Proceedings**

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| **ISSN:** | 0146-4124 |

# A NULL PSEUDOHOMOTOPIC MAP ONTO A PSEUDO-ARC

**David P. Bellamy**[1]

Problem 31 of the University of Houston Problem Book, posed by W. Kuperberg, asks whether the pseudo-arc is pseudo-contractible. This question remains open, but it will be shown here that there is a null pseudohomotopic map from a Cantor set onto a pseudo arc. The existence of such a map is quite some distance from settling the question of pseudocontractibility, but by sidestepping some of the difficulties involving connectedness, it shows that some natural simple approaches are not powerful enough to prove non-pseudocontractibility.

A *continuum* is a compact, connected metric space. A continuum is *indecomposable* if it is not the union of two of its proper subcontinua, and is *hereditarily indecomposable* if each of its subcontinua is indecomposable. If a and b are points of an indecomposable continuum X, then a and b are in the same *composant* of X provided some proper subcontinuum of X contains both a and b. This is easily seen to be an equivalence relation, and each equivalence class (composant) is a first category $F_\sigma$ set. Thus, the complement of any composant is a dense $G_\delta$.

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[1]This research was supported in part by NSF grant number MCS 8302176.

A continuum X is *arc-like* provided that for every $\varepsilon > 0$ there is a map $f: X \to [0,1]$ with each $f^{-1}(t)$ having diameter less than $\varepsilon$. A pseudo-arc is an arc-like continuum which is hereditarily indecomposable. Pseudo-arcs have been extensively studied; the information which will be needed on them here is summarized in the following lemmas, drawn from other sources as indicated.

*Lemma* 1. *Each nondegenerate subcontinuum of a pseudo-arc is a pseudo-arc, and all pseudo-arcs are homeomorphic.* [2] and [7]

*Lemma* 2. *The pseudo-arc is homogeneous; furthermore, if* a, b, c, *and* d *are points of a pseudo arc* P; a *and* b *lie in different composants of* P; *and* c *and* d *lie in different composants of* P, *then there is a homeomorphism* $h: P \to P$ *such that* $h(a) = c$ *and* $h(b) = d$. [8] or [3]

*Lemma* 3. *Every subcontinuum of a pseudo-arc* P *is a retract of* P. [4]

*Lemma* 4. *The homeomorphism group of* P *contains no nondegenerate continuum.* [6]

Let X and Y be compact metric spaces and let $f,g: X \to Y$ be continuous maps. A *pseudohomotopy* from f to g is a four-tuple $(H,Z,a,b)$, where Z is a continuum; $a,b \in Z$; $H: X \times Z \to Y$ is continuous; and, for every $x \in X$, $H(x,a) = f(x)$ while $H(x,b) = g(x)$. Z is called the *parameter space* of the pseudohomotopy. A pseudohomotopy reduces to a homotopy in case $Z = [0,1]$, $a = 0$, and $b = 1$.

A compact metric space $X$ is *pseudocontractible* provided the identity on $X$ is pseudohomotopic to a constant map from $X$ to itself.

The next two lemmas are a form of Effros' theorem and a result about open maps of complete metric spaces.

*Lemma* 5. *Suppose* $G$ *is a complete, separable, metric group acting transitively on a complete separable metric space* $X$, *and* $p \in X$. *Define* $e: G \to X$ *by* $e(g) = g(b)$. *Then* $e$ *is a continuous open mapping.* [5]

*Lemma* 6. *Let* $S$ *and* $X$ *be complete metric spaces and let* $H: S \to X$ *be a continuous open mapping. Suppose* $U$ *is open in* $S$, $L$ *is a compact subset of* $X$, *and* $L \subseteq H(U)$. *Then there is a compact set* $K \subseteq U$ *such that* $H(K) = L$. [1]

The machinery is now set up to prove the principal result:

*Theorem. There is a map from a Cantor set* $C$ *onto a pseudo-arc* $P$ *which is pseudo-homotopic to a constant map, with* $P$ *itself as parameter space.*

*Proof.* Let $P$ be a pseudo-arc and let $q \in P$. Let $D$ denote the composant of $P$ containing $q$, and let $W$ be a non-degenerate subcontinuum of $P - D$. Let $r: P \to W$ be a retraction, and define $p = r(q)$. Let $h: W \to P$ be a homeomorphism with $h(p) = q$.

Now let $G$ be the group of homeomorphisms $g$ of $P$ such that $g(q) = q$. $G$ is closed in the full homeomorphism group of $P$ and thus is completely metrizable. By Lemma 2,

$G(p) = P - D$, and since $P - D$ is a $G_\delta$ set, it is also completely metrizable. Hence, by Lemma 5, the map $e\colon G \to P - D$, defined by $e(g) = g(p)$, is an open mapping, and by Lemma 6, there is a compact set $C_0 \subseteq G$ such that $e(C_0) = W$. If $C = \{g \in C_0 \mid g$ has no countable neighborhood in $C_0\}$, then $C$ contains no isolated points, and so by Lemma 4 is topologically a Cantor set. Clearly, $e(C) = W$ also. Define $f\colon C \to P$ by $f(g) = h \circ g(p)$. Then $f$ is onto since $f(C) = h(e(C)) = h(W) = P$. Define $H\colon C \times P \to P$ by $H(g,x) = h \circ r \circ g(x)$ then $H(g,p) = h \circ r \circ g(p) = h(g(p)) = f(g)$; and $H(g,q) = h \circ r \circ g(q) = h \circ r(q) = h(p) = q$. Thus, $(H,P,p,q)$ is a pseudohomotopy from $f$ to the constant map at $q$, and the proof is done.

A couple of new questions seem appropriate to raise here.

1. Is every map from a Cantor set onto a pseudo-arc null-pseudohomotopic?

2. Is there a null-pseudohomotopic map from some continuum $X$ onto $P$?

I would like to thank Janusz M. Lysko for several interesting conversations on this problem.

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