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by

LAURENCE BOXER

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Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
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COMPRESSED COMPACTA AND SARI MAPS

Laurence Boxer[1]

1. Introduction

The notion of a compressed compactum was introduced in [Bx-S]. Here we observe that the compressed compacta coincide with the AWNR's [Bg]. We give a generalization of compression modelled on Čerin's generalizations of triviality and movability (see [C-S]). This generalization is a hereditary shape property, but in some respects does not behave as do Čerin's analogs.

In the spirit of Čerin's equicontinuous shape theory [C3] we give a "controlled" version of compression that yields a characterization of ANR's and helps us obtain a condition necessary and sufficient for ARI maps to preserve ANR's.

2. Preliminaries

Recall the following definitions:

(2.1) [Bg, p. 97] A compactum X is an *absolute weak neighborhood retract* (AWNR) if for some (indeed, for any) compact ANR Y containing X there is a neighborhood U of X in Y such that for every neighborhood V of X in Y there is a map $r: U \to V$ such that $r(x) = x$ for all $x \in X$.

(2.2) [Bx-S, p. 851] Let U be a neighborhood of a compactum X in a space Y. Then U *compresses toward* X *in*

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Y if for each neighborhood V of X in Y there exist a map $r\colon U \to V$ and a neighborhood W of X, $W \subset V$, such that $r(x) = x$ for all $x \in W$.

Let X be a compactum, C a family of metric spaces, and M an ANR-space containing X. We say X is C-*compressed in* M if there is a neighborhood U of X in M satisfying

C-comp(U,X): for every neighborhood V of X in U there is a neighborhood W of X in V such that for every C-map $f\colon K \to U$ (by C-map we mean a map whose domain K belongs to C) there is a map $g\colon K \to V$ such that $g|f^{-1}(W) = f|f^{-1}(W)$.

Note if U' is a neighborhood of X in U and C-comp(U,X), then C-comp(U',X).

The following shows the choice of M above is insignificant. The notation $f \simeq_V g$ will mean maps f and g are homotopic in V.

(2.3) *Theorem. Let* M *and* N *be* ANR-*spaces containing homeomorphic compacta* X *and* X', *respectively. If* X *is* C-*compressed in* M *then* X' *is* C-*compressed in* N.

Proof. Let $h\colon X \to X'$ be a homeomorphism. There exist neighborhoods U of X in M, U' of X' in N, and extensions $f\colon U \to N$ of h and $f'\colon U' \to M$ of h^{-1} such that C-comp(U,X) and $f^{-1}(U') \cup f'(U') \subset U$.

Let V' be a neighborhood of X' in U'. Let $V = f^{-1}(V')$. Then V is a neighborhood of X in U. Since C-comp(U,X), there is a neighborhood W of X in V such that for every C-map $F\colon K \to U$ there is a map $G\colon K \to V$ with $G|F^{-1}(W) = F|F^{-1}(W)$. Recalling our choices of f and f' and using

[H, IV 1.1] and the fact that V' is an ANR-space, there is a closed neighborhood W' of X' in V' such that $f'(W') \subset W$ and $ff'|W' \cong i_{W',V'}$ (the inclusion of W' into V').

Suppose $p: K \to U'$ is a C-map. Then $f'p: K \to U$. Our choice of W implies that there is a map $G: K \to V$ such that $G|(f'p)^{-1}(W) = f'p|(f'p)^{-1}(W)$. Our choice of W' implies $G|p^{-1}(W') = f'p|p^{-1}(W')$. Then $fG(K) \subset V'$ with $fG|p^{-1}(W') = ff'p|p^{-1}(W') \cong_{V'} p|p^{-1}(W')$ by our choice of W'. Since $p^{-1}(W')$ is closed in K and $fG|p^{-1}(W')$ extends to $fG: K \to V'$, Borsuk's Homotopy Extension Theorem [Bk, IV(8.1), p. 94] implies $p|p^{-1}(W')$ extends to $P: K \to V'$. It follows that C-comp(U',X').

In light of (2.3), we drop "in M" and say X is C-compressed if for some (hence every) ANR-space M containing X, X is C-compressed in M.

We will use the following well-known property several times:

(2.4) *Theorem* [Bk, V(3.1)]. *Let* Y *be a compact* ANR, $\varepsilon > 0$. *There is a* $\delta > 0$ *such that if* $f,g: X_0 \to Y$ *are* δ*-close maps of a closed subset* X_0 *of a metric space* X *into* Y, *and* f *extends to* $F: X \to Y$, *then there is an extension* $G: X \to Y$ *of* g *such that* F *and* G *are* ε*-close.*

3. Some Properties of C-Compressed Compacta

If X is a C-compressed compactum with C the class of all compact ANR's, we will say X is compressed. We have:

(3.1) *Theorem. Let* X *be a compactum in the Hilbert cube* Q. *The following are equivalent:*

a) X *is compressed.*

b) *There is a neighborhood* U *of* X *in* Q *such that* U *compresses toward* X *in* Q.

c) $X \in$ AWNR.

Proof. a) implies b): If X is compressed, there is a compact ANR neighborhood U of X in Q such that ANR-comp(U,X). Let V be any neighborhood of X in U. Then there is a neighborhood W of X in V such that for any ANR-map $f: K \to U$ there is a map $g: K \to V$ with $g|f^{-1}(W) = f|f^{-1}(W)$. In particular, for $1_U: U \to U$ there is a map $g: U \to V$ such that for $x \in 1_U^{-1}(W) = W$, $g(x) = 1_U(x) = x$. Hence U compresses toward X in Q.

b) implies c): This follows from (2.1) and (2.2).

c) implies a): If $X \in$ AWNR, there is a neighborhood U of X in Q such that for every neighborhood V of X in U there is a map $r: U \to V$ such that $r|X = i_{X,V}$. We may assume V is a compact ANR. By (2.4), there is a $\delta > 0$ such that if $f,g: Y_o \to V$ are δ-close maps and Y_o is a closed subset of a metric space Y, then if f extends to $F: Y \to V$ then g also extends to a map $G: Y \to V$. There is a closed neighborhood W of X in V such that $r|W$ and $i_{W,V}$ are δ-close.

Let $f: K \to U$ be an ANR-map. Then $rf: K \to V$ and (letting d^s denote the sup-metric for maps from K into Q) $d^s(rf|f^{-1}(W), f|f^{-1}(W)) < \delta$. Our choices of δ and W imply $f|f^{-1}(W)$ extends to $F: K \to V$. It follows that ANR-comp(U,X), and from (2.3) that X is compressed.

From (3.1) it follows that the following is a generalization of Tsuda's result [T, Thm. 1.2] that AWNR is a hereditary shape property.

(3.2) *Theorem. Let* X *and* Y *be compacta,* C *a class of metric spaces. If* X *is* C-*compressed and* Sh X $\geq$ Sh Y, *then* Y *is* C-*compressed.*

Proof. There is no loss of generality in assuming $X \cup Y \subset Q$. Let U be a neighborhood of X in Q such that C-comp(U,X). Let $\underline{f} = \{f_k, X, Y\}_{Q,Q}$ and $\underline{g} = \{g_k, Y, X\}_{Q,Q}$ be fundamental sequences such that $\underline{fg} \cong \underline{1}_Y$. There exist a positive integer k_1 and a neighborhood U' of Y in Q such that $k > k_1$ implies $g_k(U') \subset U$.

Let V' be any neighborhood of Y in U'. There exist a positive integer k_2 and a neighborhood V of X in U such that $k > k_2$ implies $f_k(V) \subset V'$. Our choice of U implies there is a closed neighborhood W of X in V such that for each C-map f: $K \to U$ there is a map g: $K \to V$ such that $g|f^{-1}(W) = f|f^{-1}(W)$. There exist a positive integer k_3 and a closed neighborhood W' of Y in Q with $W' \subset V'$ such that $k > k_3$ implies $g_k(W') \subset W$ and $f_k g_k|W' \cong i_{W',V'}$.

Fix $k > \max\{k_1, k_2, k_3\}$. Let h: $K \to U'$ be a C-map. Then $g_k h$: $K \to U$, so there is a map H: $K \to V$ such that $H|(g_k h)^{-1}(W) = g_k h|(g_k h)^{-1}(W)$. Our choices of k_3 and W' imply $H|h^{-1}(W') = g_k h|h^{-1}(W')$. Then $f_k H$: $K \to V'$ with $f_k H|h^{-1}(W') = f_k g_k h|h^{-1}(W') \cong_{V'} h|h^{-1}(W')$. Since V' is an ANR-space and $h^{-1}(W)$ is closed in K, we may apply Borsuk's Homotopy Extension Theorem to conclude that since $f_k H|h^{-1}(W')$ extends to

$f_k H$: $K \to V$, therefore $h|h^{-1}(W)$ extends to F: $K \to V$. It follows that C-comp(U',Y).

Recall a compactum X is C-*calm* [Cl] if for some (any) ANR-space M containing X there is a neighborhood U of X in M such that for every neighborhood V of X in U there is a neighborhood W of X in V such that if f,g: $K \to W$ are C-maps with $f \simeq_U g$, then $f \simeq_V g$.

Let us say a class C is I-closed (I is the interval [0,1]) if $K \in C$ implies $K \times I \in C$.

(3.3) *Theorem. Let* C *be an* I-*closed class of metric spaces and let* X *be a* C-*compressed compactum. Then* X *is* C-*calm.*

Proof. Let M be an ANR-space containing X and let U be a neighborhood of X in M such that C-comp(U,X). Let V be a neighborhood of X in U. Let W be a neighborhood of X in V such that for every C-map f: $K \to U$ there is a map g: $K \to V$ with $g|f^{-1}(W) = f|f^{-1}(W)$.

Suppose f_0, f_1 : $K \to W$ are C-maps with $f_0 \simeq_U f_1$. There is a map F: $K \times I \to U$ such that

$$F(x,0) = f_0(x),$$
$$F(x,1) = f_1(x)$$

for all $x \in K$. Since C is I-closed, F is a C-map. Our choice of W implies there is a map G: $K \times I \to V$ such that $G|F^{-1}(W) = F|F^{-1}(W)$. Thus $G(x,t) = F(x,t) = f_t(x)$ for $t \in \{0,1\}$, so $f_0 \simeq_V f_1$, via G. It follows that X is C-calm.

(3.4) *Corollary. If* C *is an* I-*closed class of connected metric spaces, the compactum* X *is* C-*compressed if and only if* X *has finitely many components, each of which is* C-*compressed.*

Proof. "If" is elementary. "Only if": that X C-compressed implies X has finitely many components follows from (3.3) and the fact that C-calm compacta have finitely many components [Cl, (4.6)]. Since C consists of connected spaces, it follows easily that each component of X is C-compressed.

(3.5) *Remark.* The importance of the assumption that C be I-closed in (3.3) and (3.4) is illustrated by the following: It is easily seen that every compactum is S-compressed, where S is the class consisting of one member, a space with one point; but compacta with infinitely many components are not S-calm.

(3.6) *Remark.* Let C and D be classes of compacta. We say C shape-dominates D if for each $Y \in D$ there is an $X \in C$ such that $\mathrm{Sh}\, X \geq \mathrm{Sh}\, Y$. Čerin has studied several shape properties such that if a compactum X has the property for the class C, and C shape-dominates D, then X has the property for D. This is not the case for compression: let T be the class of compacta with trivial shape. Then S shape dominates T. If X is a compactum with infinitely many components, then X is S-compressed, but not T-compressed, by (3.4).

(3.7) *Remark.* One sees easily from (3.1) that every member of T is compressed. However, in general not every C-trivial [C-S] compactum is C-compressed. If X is a compactum with infinitely many components, then X is T-trivial [Bxl, (4.3)], but not T-compressed (as seen in (3.6)).

Since solenoids are calm [C1], the following shows that the compressed compacta form a proper subclass of the class of calm compacta.

(3.8) *Example. No solenoid is compressed.*

Proof. Let X be a solenoid. If X is compressed then the suspension SX of X has the shape of a finite CW complex [T, Thm. 2.6]. Therefore $H_2(SX)$ would be movable in the category of groups, a contradiction.

The author is indebted to an anonymous colleague for suggesting the proof above in place of the author's original, longer proof.

A continuous surjection of compacta $f: X \to Y$ is called *approximately right invertible* (ARI) [G, p. 293] if there is a null sequence of positive numbers ε_n such that for each n there is a map $g_n: Y \to X$ with $d^s(fg_n, 1_Y) < \varepsilon_n$.

If there are AR-spaces M and N containing X and Y, respectively, such that the maps g_n above can be extended to $G_n: N \to M$ such that for every neighborhood U of X in M there is a neighborhood V of Y in N with $G_n(V) \subset U$ for almost all n, then f is called *strongly approximately right invertible* (SARI) [C2]. The choices of M and N are not significant.

It is known that ARI and SARI maps preserve a number of shape and e-shape invariants ([Bx2], [Bx3], [C2]).

We have:

(3.9) *Theorem. Let* X *and* Y *be compacta,* $f: X \to Y$ *an* SARI *map. If* X *is* C*-compressed then* Y *is* C*-compressed.*

Proof. We may assume $X \cup Y \subset Q$. Let U be a neighborhood of X in Q such that C-comp(U,X). Since $f \in$ SARI, there is a sequence $\{G_n: (Q,Y) \to (Q,X)\}$ of maps of pairs as described above. There exist a positive integer n_1 and a neighborhood U' of Y in Q such that $n > n_1$ implies $G_n(U') \subset U$. Let $F: Q \to Q$ extend f.

Let V' be any compact ANR neighborhood of Y in U'. There is a neighborhood V of X in U such that $F(V) \subset V'$.

By (2.4), there is an $\varepsilon > 0$ such that if $h_1, h_2: Z_0 \to V'$ are ε-close maps of a closed subset Z_0 of a metric space Z into V' and h_1 extends to $H_1: Z \to V'$, then h_2 extends to $H_2: Z \to V'$. Let W be a closed neighborhood of X in V such that for each C-map $u: K \to U$ there is a map $u_1: K \to V$ with $u_1|u^{-1}(W) = u|u^{-1}(W)$. There exist a positive integer n_2 and a closed neighborhood W' of Y in V' such that $n > n_2$ implies $G_n(W') \subset W$ and $FG_n|W'$ is ε-close to $i_{W',V'}$.

Fix $n > \max\{n_1, n_2\}$. Let $r: K \to U'$ be a C-map. Then $G_n r: K \to U$. Our choice of W implies there is a map $s: K \to V$ such that $s|(G_n r)^{-1}(W) = G_n r|(G_n r)^{-1}(W)$. By choice of W', $s|r^{-1}(W') = G_n r|r^{-1}(W')$. By choice of V, $Fs: K \to V'$ and $Fs|r^{-1}(W') = FG_n r|r^{-1}(W')$. Since (by choices of W' and n_2) $FG_n r|r^{-1}(W')$ is ε-close to $r|r^{-1}(W')$, and since $Fs|r^{-1}(W')$ extends to $Fs: K \to V'$, it follows from our choice of ε that $r|r^{-1}(W')$ extends to $R: K \to V'$. It follows that C-comp(U',Y).

If $f: X \to Y$ is an SARI map of compacta such that (using notation as above) there is a sequence of maps of pairs $\{h_n: (M,X) \to (N,Y)\}$ such that $d^s(g_n h_n | X, 1_X) < \varepsilon_n$, each $h_n(X) = Y$ for all n, and for every neighborhood U of Y in N there is a neighborhood V of X in M such that $h_n(V) \subset U$ for almost all n, then f is called *strongly approximately invertible* (SAI) [C2]. We have:

(3.10) *Theorem. Let* $f: X \to Y$ *be an* SAI *map of compacta. Then* X *is* C-*compressed if and only if* Y *is* C-*compressed.*

Proof. If X is C-compressed then so is Y, by (3.9), since f is SARI.

Suppose Y is C-compressed. Assume that $M = N = Q$ and let $F: Q \to Q$ extend f. Let $\{G_n: (Q,Y) \to (Q,X)\}$, $\{h_n: (Q,X) \to (Q,Y)\}$ be as above. Let U' be a neighborhood of Y in Q such that C-comp(U',Y). There exist a positive integer n_1 and a neighborhood U of X in Q such that $n > n_1$ implies $h_n(U) \subset U'$.

Let V be a compact ANR neighborhood of X in U. There exist a positive integer n_2 and a neighborhood V' of Y in U' such that $n > n_2$ implies $G_n(V') \subset V$. By choice of U', there is a neighborhood W' of Y in V' such that for each C-map $b: K \to U'$ there is a map $c: K \to V'$ with $c|b^{-1}(W') = b|b^{-1}(W')$. By (2.4), there is an $\varepsilon > 0$ such that if $u_1, u_2: Z_0 \to V$ are ε-close maps of a closed subset Z_0 of a metric space Z into V and u_1 extends to $U_1: Z \to V$, then u_2 extends to $U_2: Z \to V$.

There exist a neighborhood W_0 of X in V and a positive integer n_3 such that $n > n_3$ implies $G_n h_n|X$ is ε-close to $i_{X,V}$ and $h_n(W_0) \subset W'$.

Fix $n > \max\{n_1,n_2,n_3\}$. There is a closed neighborhood W of X in W_0 such that $G_n h_n|W$ and $i_{W,V}$ are ε-close.

Let $s: K \to U$ be a C-map. Then $h_n s: K \to U'$. Hence there is a map $c: K \to V'$ such that $c|(h_n s)^{-1}(W') = h_n s|(h_n s)^{-1}(W')$. It follows that $c|s^{-1}(W) = h_n s|s^{-1}(W)$. Therefore $G_n c: K \to V$ with $G_n c|s^{-1}(W) = G_n h_n s|s^{-1}(W)$. But $G_n h_n s|s^{-1}(W)$ is ε-close to $s|s^{-1}(W)$, and since the former extends to $G_n c: K \to V$ and $s^{-1}(W)$ is closed in K, therefore $s|s^{-1}(W)$ extends (by choice of ε) to $t: K \to V$. It follows that C-comp(U,X).

4. C-e-Compressed Compacta

As Čerin has observed, various open questions suggest the need for further characterizations of ANR's. For example, we find in [G, p. 294]:

(4.0) Do ARI maps or refinable maps [F-R] preserve ANR's?

(Partial answers are in [F-K] and [C2].) In this section we give a characterization of ANR's in terms of an e-shape version of compression and show how we may better understand (4.0). For the sake of convenience, we assume in this section that X and Y are compacta in Q.

For a family C of metric spaces, we say X is C-e-*compressed* if for every $\varepsilon > 0$ there is a neighborhood U of X in Q such that C^{ε}-comp(U,X): for every neighborhood V of X in U there is a neighborhood W of X in V such that for every C-map $f: K \to U$ there is a map $g: K \to V$ with $g|f^{-1}(W) = f|f^{-1}(W)$ and $d^s(f,g) < \varepsilon$.

It is easily seen that if U_1 is a neighborhood of X in U and C^ε-comp(U,X), then C^ε-comp(U_1,X). Also, every compactum is S-e-compressed.

If X is C-e-compressed when C is the family of compact ANR's, we say X is e-compressed.

We give a series of results relating C-e-compression to the e-shape properties C-e-movability [C3] and C-e-calmness [C2].

(4.1) *Theorem. A* C-e-*compressed compactum is* C-e-*movable.*

Proof. Let U be a neighborhood of the C-e-compressed compactum X in Q and let $\varepsilon > 0$. We may assume U is a compact ANR. There is a $\delta > 0$ such that δ-close maps into U are ε-homotopic in U. There is a neighborhood V of X in U such that C^δ-comp(V,X). Thus, for every neighborhood W of X in V and every C-map $f: K \to V$ there is a map $g: K \to W$ with $d^s(f,g) < \delta$. By choice of δ, f and g are ε-homotopic in U. It follows that X is C-e-movable.

(4.2) *Theorem. Let* C *be an* I-*closed class of metric spaces and let* X *be* C-e-*compressed. Then* X *is* C-e-*calm.*

Proof. Let $\varepsilon > 0$. There is a compact ANR neighborhood U of X in Q such that $C^{\varepsilon/3}$-comp(U,X). Let $\delta > 0$ be such that δ-close maps into U are $(\varepsilon/3)$-homotopic.

Let V be a neighborhood of X in U. By choice of U, there is a neighborhood W of X in V such that for every

C-map $f: K \to U$ there is a map $g: K \to V$ such that $g|f^{-1}(W) = f|f^{-1}(W)$ and $d^s(f,g) < \varepsilon/3$.

Suppose $h_0,h_1: K \to W$ are δ-close. By choice of δ, there is a map $F: K \times I \to U$ such that for all $x \in K$,

$$F(x,0) = h_0(x),$$
$$F(x,1) = h_1(x) \text{ and}$$
$$\operatorname{diam} F(\{x\} \times I) < \varepsilon/3.$$

Since $K \times I \in C$, there is a map $G: K \times I \to V$ such that $G|F^{-1}(W) = F|F^{-1}(W)$ and $d^s(F,G) < \varepsilon/3$.

For $x \in K$, it follows that $G(x,0) = h_0(x)$, $G(x,1) = h_1(x)$, and for all $s,t \in I$, $d(G(x,s),G(x,t)) \leq d(G(x,s),F(x,s)) + d(F(x,s),F(x,t)) + d(F(x,t),G(x,t)) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$. Thus h_0 and h_1 are ε-homotopic in V. It follows from [C2, (4.2)] that X is C-e-calm.

(4.3) *Theorem.* $X \in$ ANR *if and only if* X *is e-compressed.*

Proof. Suppose $X \in$ ANR. Let $\varepsilon > 0$. There is a compact ANR neighborhood U of X in Q and a retraction $r: U \to X$ such that $d^s(r,1_U) < \varepsilon/2$. Let V be a neighborhood of X in U. We may assume V is a compact ANR. Then there is, by (2.4), a $\delta > 0$ such that if Z_0 is a closed subset of a metric space Z, h_0 and h_1 are δ-close maps of Z_0 into V, and h_0 extends to $H_0: Z \to V$, then there is an extension $H_1: Z \to V$ of h_1, such that $d^s(H_0,H_1) < \varepsilon/2$. There is a closed neighborhood W of X in V such that $d^s(r|W,i_{W,V}) < \delta$.

Let $f: K \to U$ be an ANR-map. We have $d^s(rf|f^{-1}(W), f|f(W)) < \delta$. Since $f^{-1}(W)$ is closed in K and $rf|f^{-1}(W)$ extends to $rf: K \to X \subset V$, our choice of δ implies $f|f^{-1}(W)$

extends to a map $g\colon K \to V$ with $d^s(rf,g) < \varepsilon/2$. Hence $d^s(f,g) \leq d^s(f,rf) + d^s(rf,g) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. It follows that $\mathrm{ANR}^{\varepsilon}$-comp$(U,X)$, and thus X is e-compressed.

Conversely, if X is e-compressed then by (4.1) and (4.2), X is e-movable and e-calm. Hence $X \in$ ANR, by [C2, (4.9a)].

Let $f\colon X \to Y$ be an SARI map and let $F\colon Q \to Q$ extend f. Let $\{G_n\colon (Q,Y) \to (Q,X)\}_{n=1}^{\infty}$ be a sequence of maps of pairs as in the previous section. We will say f is *extra strongly approximately right invertible* (ESARI) if for every $\varepsilon > 0$ there is a neighborhood U of Y in Q such that $d^s(FG_n|U, i_{U,Q}) < \varepsilon$ for almost all n. One sees easily that the choice of extension F of f is not important.

The following is suggested by [C2, (5.1)].

(4.4) *Theorem. Let* $Y \in$ ANR *and suppose* $f\colon X \to Y$ *is an* ARI *map. Then* f *is an* ESARI *map.*

Proof. We assume $F\colon Q \to Q$ extends f.

Since f is an ARI map, there is a sequence of maps $g_n\colon Y \to X$ such that $fg_n \xrightarrow[n\to\infty]{} 1_Y$. Let Z be a neighborhood of Y in Q such that there is a retraction $r\colon Z \to Y$. Let $G_n\colon Q \to Q$ be an extension of $g_n r\colon Z \to X$.

Then for each n, given a neighborhood U of X in Q, Z is a neighborhood of Y in Q such that $G_n(Z) \subset U$. It follows easily that f is an SARI map. Further, let $\varepsilon > 0$ be given. There is a neighborhood V of Y in Z such that $d^s(r|V, i_{V,Z}) < \varepsilon/2$ and there is a positive integer z such that $n > z$ implies $d^s(fg_n, 1_Y) < \varepsilon/2$. Hence for $n > z$,

$d^s(FG_n|V, i_{V,Q}) \leq d^s(FG_n|V, r|V) + d^s(r|V, i_{V,Q}) < d^s(fg_n r|V, r|V) + \varepsilon/2 < \varepsilon$. It follows that f is ESARI.

The following, suggested by [C2, (5.2) and (5.3)], gives a converse to (4.4) when it is assumed $X \in$ ANR.

(4.5) *Theorem. Let* $X \in$ ANR *and suppose* $f: X \to Y$ *is* ESARI. *Then* $Y \in$ ANR.

Proof. By (4.3), it suffices to show Y is e-compressed. Let $F: Q \to Q$ extend f and let $\{G_n: (Q,Y) \to (Q,X)\}_{n=1}^{\infty}$ be a sequence of maps of pairs satisfying the definition of ESARI.

Let $\varepsilon > 0$. Since $X \in$ ANR, there is a neighborhood U' of X in Q and a retraction $r: U' \to X$ such that $d^s(F|U', Fr) < \varepsilon/3$. Since f is ESARI, there exist a neighborhood U of Y in Q and a positive integer n_1 such that $n > n_1$ implies

$$G_n(U) \subset U' \text{ and}$$
$$d^s(FG_n|U, i_{U,Q}) < \varepsilon/3.$$

Let V be a neighborhood of Y in U. We may assume V is a compact ANR. By (2.4), there is a $\delta > 0$ such that if $h_0, h_1: Z_0 \to V$ are δ-close maps of a closed subset Z_0 of a metric space Z into V and h_0 extends to $H_0: Z \to V$, then h_1 extends to $H_1: Z \to V$ with $d^s(H_0, H_1) < \varepsilon/3$. There is a neighborhood W' of X in U' such that $d^s(F|W', fr|W') < \delta/2$. There exist a closed neighborhood W of Y in V and a positive integer n_2 such that $n > n_2$ implies

$$G_n(W) \subset W',$$
$$FG_n(W) \subset V, \text{ and}$$
$$d^s(FG_n|W, i_{W,V}) < \delta/2.$$

Fix $n > \max\{n_1, n_2\}$. Let $q: K \to U$ be an ANR-map. Then $G_n q: K \to U'$, $rG_n q: K \to X$, and we define $b = frG_n q: K \to Y \subset V$. Note $d^s(b|q^{-1}(W), q|q^{-1}(W)) \leq d^s(frG_n q|q^{-1}(W), FG_n q|q^{-1}(W)) + d^s(FG_n q|q^{-1}(W), q|q^{-1}(W)) < \delta/2 + \delta/2 = \delta$. Since $q^{-1}(W)$ is closed in K and $b|q^{-1}(W)$ extends to $b: K \to V$, our choice of δ implies $q|q^{-1}(W)$ extends to $g: K \to V$ with $d^s(b,g) < \varepsilon/3$. Thus $d^s(q,g) \leq d^s(q,b) + d^s(b,g) < d^s(q, FG_n q) + d^s(FG_n q, frG_n q) + \varepsilon/3 <$ (by our choices of U and U') $\varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$. Thus ANR^{ε}-comp(U,Y), and the proof is complete.

It follows from [F-R, (3.3)] that a refinable ANR-map is ARI. In light of (4.4) and (4.5), it follows that the questions (4.0) are equivalent to the following:

(4.7) *Questions.* If $f: X \to Y$ is ARI (refinable) and $X \in$ ANR, must f be ESARI?

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Niagara University

Niagara University, New York 14109