A LOCALLY COMPACT, HOMOGENEOUS METRIC SPACE WHICH IS NOT BIHOMOGENEOUS

by

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Suppose $S$ is a topological space. If, for each two points $a$ and $b$ of $S$, there is a homeomorphism of $S$ onto $S$ throwing $a$ to $b$, then $S$ is said to be homogeneous. If, for each two points $a$ and $b$ of $S$, there is a homeomorphism of $S$ onto $S$ throwing $a$ to $b$ and $b$ to $a$, then $S$ is said to be bihomogeneous. In [2], Kuratowski gave an example of a connected metric homogeneous space which is not bihomogeneous. His example is not locally compact (nor, indeed, is it topologically complete). Here, we give a locally compact example.

The Space $X$. Let $P$ denote a pseudo-arc and $C(P)$ denote the space of all subcontinua of $P$. Let $X$ denote the subspace of $C(P)$ comprising the nondegenerate proper subcontinua of $P$.

Theorem 0. The space $X$ is connected and locally compact.

Proof. There is a continuous collection $H$ of proper subcontinua of $P$ filling up $P$, [1]. Now, $H$ is a continuum in $X$ and each point of $X$ is connected to $H$ by an arc, [1]. Thus, $X$ is connected.

Since $X$ is an open subset of the compact space $C(P)$, $X$ is locally compact.
Theorem 1. The space $X$ is homogeneous.

Proof. Lehner has shown [4] that, if $a$ and $b$ are two proper subcontinua of $P$, then there is a homeomorphism $h$ of $P$ onto $P$ that takes $a$ onto $b$. Then $h^*$, the homeomorphism of $C(P)$ onto $C(P)$ induced by $h$, throws $X$ onto $X$ and $h^*(a) = b$. Thus, $X$ is homogeneous.

Theorem 2. The space $X$ is not bihomogeneous.

Proof. Let $a$ and $b$ be two points of $X$ such that $b$ is a proper subcontinuum of $a$. Further, let $b_1, b_2, b_3, \cdots$ be a sequence of proper subcontinua of $a$ (points of $X$) converging to $b$ such that no two of them lie in the same composant of $a$ and no one of them lies in the composant of $a$ containing $b$. There is only one arc $ab$ in $X$ from $a$ to $b$; for each $i$, there is only one arc $ab$, from $a$ to $b_i$; and if $tb$ is an arc in $X$ from a point $t \neq b$ of $X$ to $b$ such that $tb \cap ab = b$ then $t$ is a proper subcontinuum of $b$ (in $P$), [1].

Now, suppose that $h$ is a homeomorphism from $X$ onto $X$ such that $h(a) = b$ and $h(b) = a$. For each $i$, let $t_i = h(b_i)$. Now, $h$ throws the arc $ab$ onto itself, and, for each $i$, throws the arc $ab_i$ onto an arc $t_i b$ such that $t_i b \cap ab = b$. Then, for each $i$, $t_i$ is a proper subcontinuum (in $P$) of $b$. But the sequence $t_1, t_2, t_3, \cdots$ converges to $h(b) = a$. But no sequence of proper subcontinua of $b$ converges to $a$.

Thus, $X$ is not bihomogeneous.

Notes. For a general exposition of the pseudo-arc, see [3]. For a thorough discussion of hyperspaces such as $C(P)$, see [5].
Kuratowski's example, mentioned in the opening paragraph, is a 1-dimensional subspace of the plane. The space $X$ is a 2-dimensional subspace of Euclidean 3-space, [6]. K. Kuperberg tells me that she has an example (to appear) of a 3-dimensional compact metric continuum which is homogeneous but not bihomogeneous.

References


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