A HYPERSPACE RETRACTION THEOREM FOR A CLASS OF HALF-LINE COMPACTIFICATIONS

by

D. W. CURTIS
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1. Hyperspace Retractions

For X a metric continuum, let $2^X$ be the hyperspace of all nonempty subcompacta, with the Hausdorff metric topology, and let $C(X) \subseteq 2^X$ be the hyperspace of subcontinua. If X is locally connected, both $C(X)$ and $2^X$ are absolute retracts [9], and in particular $C(X)$ is a retract of $2^X$. In the non-locally connected case, neither hyperspace is an absolute retract, but we may still ask whether $C(X)$ is a retract of $2^X$. Until now, this question has been answered in only two specific cases. In 1977, Goodykoontz [2] constructed a 1-dimensional continuum X in $\mathbb{R}^3$ such that $C(X)$ is not a retract of $2^X$. And in 1983, Goodykoontz [3] showed that for X the cone over a convergent sequence, $C(X)$ is a retract of $2^X$. Thus, for X non-locally connected, $C(X)$ is not necessarily a retract of $2^X$, but it may be. (Nadler [6] had earlier shown the existence of surjections from $2^X$ to $C(X)$, in all cases.)

At present, a completely general answer for the hyperspace retraction question seems out of reach. In this paper, we answer the question for a certain class of non-locally connected continua, large enough to be of interest, but sufficiently delimited so as to be manageable. This class will consist of those half-line compactifications with locally connected remainder which are "regular" in the
following sense. Let \( X = (0, \infty) \cup K \) denote an arbitrary half-line compactification with a nondegenerate locally connected remainder \( K \) (which is therefore a Peano continuum). In this situation, there always exists a retraction \( X \twoheadrightarrow K \).

We say that \( X \) is a regular compactification if there exists a retraction \( r: X \twoheadrightarrow K \) such that, for some homeomorphism \( \phi: [0, \infty) \to [0, \infty) \), the map \( r \circ \phi: [0, \infty) \to K \) is a periodic surjection, i.e., there exists \( p > 0 \) such that \( r(\phi(t)) = r(\phi(t + p)) \) for all \( t \). Our main result is that the only regular half-line compactifications for which there exist hyperspace retractions \( 2^X \to C(X) \) are the following: the topologist's sine curve; the circle with a spiral; and a sequence of other regular compactifications with a circle as remainder, to be described below.

The case of the circle with a spiral (labelled below as \( X_1 \)) is of particular interest. It is known that \( \text{Cone } X_1 \) does not have the fixed point property \([5]\), and that \( C(X_1) \) is homeomorphic to \( \text{Cone } X_1 \) \([8]\). Noting this, Nadler \([7]\) conjectured that \( 2^X \to C(X) \) is not regular (which would make it the first such example to be known), and that the way to prove this is to construct a retraction from \( 2^X_1 \) to \( C(X_1) \). Our result confirms his conjecture.

Every periodic surjection \( \pi: [0, \infty) \to K \) onto a Peano continuum induces a regular compactification \( X(\pi) \), which may be defined as follows:

\[
X(\pi) = \{(t, \pi(t)): t \geq 0\} \cup \{(\infty, k): k \in K\} \subseteq [0, \infty) \times K.
\]
Alternatively, we may consider \(X(\pi)\) to be the disjoint union \([0,\infty) \cup K\), with the topology defined by the open base

\[
\{U: U \text{ open in } [0,\infty)\} \cup \{V \cup (\pi^{-1}(V) \cap (N,\infty)) : V \text{ open in } K \text{ and } N < \infty\}.
\]

Clearly, every regular half-line compactification is homeomorphic to some \(X(\pi)\).

Let \(I = [-1,1]\), and \(S = \{z: |z| = 1\}\), the unit circle in the complex plane. Define \(\pi_0: [0,\infty) \to I\) by \(\pi_0(t) = \sin \pi t\); define \(\pi_1: [0,\infty) \to S\) by \(\pi_1(t) = e^{i\pi t}\); and for \(n > 1\), define \(\pi_n: [0,\infty) \to S\) by the formulas

\[
\pi_n(t) = \begin{cases} 
e^{-in\pi t}, & 0 < t < 1 \pmod{2} \\ e^{in\pi t}, & 1 \leq t \leq 2 \pmod{2}. \end{cases}
\]

Then \(X_0 = X(\pi_0)\) is the topologist's sine curve; \(X_1 = X(\pi_1)\) is the circle with a spiral; and for \(n = 2,3,\ldots\), \(X_n = X(\pi_n)\) is the regular compactification obtained by alternately "wrapping" and "unwrapping" subintervals of \([0,\infty)\) about \(S\), with each subinterval covering \(S\) \(n/2\) times. Note that the spaces \(X_0, X_1, X_2, \ldots\) are topologically distinct.
Theorem. For $X$ a regular half-line compactification, there exists a hyperspace retraction $2^X \to C(X)$ if and only if $X$ is homeomorphic to some $X_n$, $n = 0, 1, 2, \ldots$.

Of course, no hyperspace retraction $2^X \to C(X)$ for non-locally connected $X$ can be quite as nice as those which may be constructed in the locally connected case. For locally connected $X$, we may use a convex metric $d$, and define a retraction $R: 2^X \to C(X)$ by taking $R(A) = \overline{N}_d(A; t)$, where $t > 0$ is the smallest value for which $\overline{N}_d(A; t) \in C(X)$. Such a retraction has the property that $R(A) \supseteq A$ for each $A \in 2^X$. Clearly, this is impossible for non-locally connected $X$.

However, there may exist a retraction $R: 2^X \to C(X)$ such that $R(A) \cap A \neq \emptyset$ for each $A$ (we say that $R$ is conservative).

In the course of proving the above theorem, it will be shown that only for $X_0$ and $X_1$ do there exist conservative hyperspace retractions.

In the final section of the paper, we note the connection between the existence of a hyperspace retraction $2^X \to C(X)$ and the existence of a mean for $C(X)$, and we give examples of continua $X$ (from the class of regular half-line compactifications) for which $C(X)$ does not admit a mean, thereby answering a question of Nadler [7].

2. A Necessary Condition

Let $X$ be any metric continuum, and let $\rho$ denote the Hausdorff metric on $2^X$. We say that $X$ has the subcontinuum approximation property if for each $\varepsilon > 0$ there exists $\delta > 0$ such that, for all $L, M \in C(X)$ with $\rho(L, M) < \delta$, and for
every subcontinuum \( P \subset M \), there exist \( P', M' \in C(X) \) with 
\[ \rho(P, P') < \epsilon, \rho(M, M') < \epsilon, \text{ and } L \cup P \subset M'. \] (In the locally connected case we may of course choose \( M' \) such that 
\( L \cup M \subset M' \), but in general \( M \) and \( M' \) will be disjoint.)

We will show that this property is a necessary condition
for the existence of a hyperspace retraction \( 2^X \to C(X) \),
and that a regular half-line compactification has the property if and only if the remainder is either an arc or a simple closed curve.

In what follows, we shall have occasion to use order arcs and segments in the hyperspaces \( 2^X \) and \( C(X) \). An arc \( a \in 2^X \) is an order arc if for each \( E, F \in a \), either \( E \subset F \) or \( F \subset E \). For elements \( A, B \in 2^X \), there exists an order arc \( a \) with \( \cap a = A \) and \( \cup a = B \) if and only if \( A \subset B \) and each component of \( B \) intersects \( A \). Every order arc \( a \) can be uniquely parametrized as a segment \( a: [0,1] \to 2^X \) with
respect to a given Whitney map \( \omega: 2^X \to [0,\infty) \), i.e.,
\[ a = \{a(t): 0 \leq t \leq 1\} \text{, with } a(0) = \cap a, \ a(1) = \cup a, \text{ and } \omega(a(t)) = (1 - t)\omega(a(0)) + t\omega(a(1)) \text{ for each } t. \] (Order arcs were first used by Borsuk and Mazurkiewicz [1] to show that \( C(X) \) and \( 2^X \) are arcwise connected. Segments were introduced by Kelley [4], who also formulated the necessary and sufficient conditions given above for the existence of an order arc, or segment, from \( A \) to \( B \).) Let
\[ \Gamma(X) = \{a \in C(2^X): a \text{ is an order arc or } a = \{A\} \text{ for } A \in 2^X\}, \]
and let \( S(\omega) \) be the function space of all segments
\[ a: [0,1] \to 2^X \] (including the constant maps), with the topology of uniform convergence. Then the spaces \( \Gamma(X) \) and
S(ω) are compact, and the natural correspondence α → \{α(t): 0 ≤ t ≤ 1\} is a homeomorphism from S(ω) to Γ(X) (for a complete discussion, see [7]). Henceforth, we implicitly use this correspondence wherever convenient. Without confusion, we let ρ denote both the Hausdorff metric on 2^X and the sup metric on S(ω).

2.1. Lemma. Let P, M ∈ C(X), with P ⊆ M. Then for each ε > 0 there exists δ > 0 such that, for every L ∈ C(X) with ρ(L, M) < δ, there exist order arcs α ∈ 2^X and β ∈ C(X) with α(1) = L, β(0) = P, β(1) = M, and ρ(α, β) < ε.

Proof. Suppose that for some ε > 0 there exists a sequence \{L_i\} in C(X) converging to M, with no L_i satisfying the required condition. Choose a finite subset F ⊆ P such that ρ(F, P) < ε. For each x ∈ F and each i, choose x_i ∈ L_i and an order arc α_{x_i} ∈ C(X) such that x_i → x, α_{x_i}(0) = \{x_i\}, and α_{x_i}(1) = L_i. Then for each i let α_i be the order arc in 2^X defined by α_i(t) = ∪{α_{x_i}(t): x ∈ F}. Thus α_i(0) = \{x_i: x ∈ F\} and α_i(1) = L_i. Since the space Γ(X) is compact, some subsequence of \{α_i\} must converge to an order arc λ in 2^X with λ(0) = F and λ(1) = M. Define an order arc β in C(X) by β(t) = P ∪ λ(t). Thus β(0) = P and β(1) = M. Since ρ(λ, β) < ε, we have ρ(α_i, β) < ε for some large i, contradicting our supposition about the sequence \{L_i\}.

2.2. Proposition. Let X be any continuum for which there exists a hyperspace retraction 2^X → C(X). Then X has the subocontinum approximation property.
Proof. Suppose $X$ does not have the property. Then by compactness of $C(X)$, there exist $P, M \in C(X)$ with $P \subseteq M$, and a sequence $\{L_i\}$ in $C(X)$ converging to $M$ such that, for some $\varepsilon > 0$, there do not exist $P', M' \in C(X)$ with $\rho(P, P') < \varepsilon$, $\rho(M, M') < \varepsilon$, and $L_i \cup P' \subseteq M'$ for some $i$. Let $R: 2^X + C(X)$ be a retraction. Choose $0 < \eta < \varepsilon$ such that, for every $A \in 2^X$ with $\rho(A, M_0) < \eta$ for some subcontinuum $M_0 \subseteq M$, $\rho(R(A), M_0) < \varepsilon$. By (2.1), for sufficiently large $i$ there exist order arcs $\alpha \subseteq 2^X$ and $\beta \subseteq C(X)$ with $\alpha(1) = L_i$, $\beta(0) = P$, $\beta(1) = M$, and $\rho(\alpha, \beta) < \eta$. Then the continua $P' = R(\alpha(0))$ and $M' = U(R(\alpha(t)) : 0 < t < 1)$ satisfy the conditions $\rho(P, P') < \varepsilon$, $\rho(M, M') < \varepsilon$, and $L_i \cup P' \subseteq M'$, contradicting our supposition.

Note. The example constructed by Goodykoontz in [2] does not have the subcontinuum approximation property; our proof for (2.2) is a generalization of his argument for the non-existence of a hyperspace retraction.

2.3. Lemma. Let $\pi: I \rightarrow K$ be a map of an arc onto a Peano continuum which is neither an arc nor a simple closed curve. Then for some subarc $J \subseteq I$, $\pi(J)$ is a proper subcontinuum of $K$ containing a simple triod.

Proof. Let $\mathcal{L}$ denote the collection of all proper subcontinua of $K$ which are of the form $\pi(J)$ for some subarc $J$. Since $K$ is neither an arc nor a simple closed curve, there must be some $L \in \mathcal{L}$ which is not an arc. Then the Peano continuum $L$ either contains a simple triod or is a simple closed curve. In either case there exists $\tilde{L} \in \mathcal{L}$ properly containing $L$, and therefore containing a simple triod.
2.4. Lemma. Let \( \pi: I \to T \) be a map of an arc onto a simple triod. Then there exists a subcontinuum \( P \subset T \) such that \( P \neq \pi(J) \) for any subarc \( J \subset I \).

Proof. Choose a sequence \( \{T_n\} \) of triods in \( T \) such that \( T_n \subset \text{int} \ T_{n+1} \). Suppose that for each \( n \) there exists a subarc \( J_n \subset I \) with \( \pi(J_n) = T_n \). We may assume that each endpoint of \( J_n \) is mapped to an endpoint of \( T_n \). Since for \( m < n \), \( T_m \subset \text{int} \ T_n \), we must have either \( J_m \cap J_n = \emptyset \) or \( J_m \subset J_n \). Choose \( \delta > 0 \) such that for each \( A \subset I \) with \( \text{diam} \ A < \delta \) and each \( n \), \( \pi(A) \) contains at most one endpoint of \( T_n \). Since one of the endpoints of \( T_n \) can be the image only of interior points of \( J_n \), it follows that \( \text{diam} \ J_n \geq 2\delta \) for each \( n \). Also, if \( m < n \) and \( J_m \subset J_n \), then \( \text{diam} \ J_n \geq \text{diam} \ J_m + \delta \). The sequence \( \{J_n\} \) in \( C(I) \) clusters at some nondegenerate \( J \). But for any pair of distinct arcs \( J_m, J_n \) sufficiently close to \( J \), it's impossible that either \( J_m \cap J_n = \emptyset \) or \( J_m \subset J_n \). Thus some \( T_n \) must satisfy the conclusion of the lemma.

2.5. Proposition. A regular half-line compactification has the subcontinuum approximation property if and only if the remainder is either an arc or a simple closed curve.

Proof. Let \( X = [0,\infty) \cup K \) be the regular half-line compactification corresponding to a periodic surjection \( \pi: [0,\infty) \to K \), and let \( I \subset [0,\infty) \) be a subarc such that \( \pi \) goes through at least two complete cycles over \( I \).

Suppose first that \( K \) is neither an arc nor a simple closed curve. Applying (2.3) to the restriction \( \pi/I \), we
obtain a proper subcontinuum $M \subset K$ such that $M$ contains a simple triod $T$ and $M = \pi(J)$ for some subarc $J \subset I$. Thus, there exists a sequence $\{J_i\}$ of subarcs in $[0,\infty)$ converging to $M$, and since $M \neq K$, every $M' \in \mathcal{C}(X)$ sufficiently close to $M$ and containing some $J_i$ must itself be a subarc of $[0,\infty)$. Let $r: K \to T$ be any retraction, and apply (2.4) to the map $r \circ \pi: I \to T$. We obtain a subcontinuum $P \subset T$ such that $P \neq \pi(I_0)$ for any subarc $I_0 \subset I$. Thus, every $P' \in \mathcal{C}(X)$ sufficiently close to $P$ must lie in $K$. It follows that $X$ does not have the subcontinuum approximation property with respect to the pair $(M,P)$.

Now suppose that $K$ is either an arc or a simple closed curve, and consider any $P,M \in \mathcal{C}(X)$ with $P \subset M$. It suffices to verify the subcontinuum approximation property with respect to this pair (see the proof of (2.2)). The property is obvious if either $M \subset [0,\infty)$ or $M \supset K$, so we may suppose that $M$ is a proper subcontinuum of $K$ (and therefore an arc). Each $L \in \mathcal{C}(K)$ which is close to $M$ intersects $M$, so in this case we may take $M' = L \cup M$ and $P' = P$. And for any arc $L \subset [0,\infty)$ close to $M$, there is a subarc $I_0 \subset L$ close to $P$, so we may take $M' = L$ and $P' = L_0$. This completes the argument that $X$ has the subcontinuum approximation property.

It may be of interest to note that the subcontinuum approximation property is implied by property $[K]$, which was introduced by Kelley [4] in the study of hyperspace contractibility and which has been used extensively in recent years (see [7]). In the class of regular half-line
compactifications, the only spaces with property \([K]\) are the spaces \(X_0\) and \(X_1\) which admit conservative hyperspace retractions. Thus, the spaces \(X_n\) for \(n > 1\) show that property \([K]\) is not necessary for the existence of hyperspace retractions. Whether there is any general relationship between property \([K]\) and the existence of conservative hyperspace retractions remains an open question.

### 3. A Monotonicity Requirement

Let \(X = [0,\infty) \cup K\) be the regular half-line compactification corresponding to a periodic surjection \(\pi: [0,\infty) \to K\), and suppose there exists a hyperspace retraction \(2^X \to C(X)\). By (2.2) and (2.5), the remainder \(K\) is either an arc or a simple closed curve. In the case that \(K\) is an arc, we say that \(\pi\) is interior monotone if, for each arc \(J \subset [0,\infty)\) such that \(\pi(J) \cap gK = \emptyset\), the restriction \(\pi/J\) is monotone (perhaps nonstrictly). A similar definition is made in the case that \(K\) is a simple closed curve, using a covering projection \((-\infty,\infty) \to K\). Specifically, let \(\tilde{\pi}: [0,\infty) \to (-\infty,\infty)\) be a lift of \(\pi\), and set \(\tilde{K} = \text{im}\ \tilde{\pi}\). We say that \(\tilde{\pi}\) is interior monotone if \(\tilde{\pi}/J\) is monotone for each arc \(J \subset [0,\infty)\) such that \(\tilde{\pi}(J) \cap \tilde{gK} = \emptyset\). We will show that \(\pi\), or \(\tilde{\pi}\), must be interior monotone. It follows easily that either \(X \approx X_0\) (if \(K\) is an arc), or \(X \approx X_1\) (if \(K\) is a simple closed curve and \(\tilde{K}\) is unbounded), or \(X \approx X_n\) for some \(n > 1\) (if \(\tilde{K}\) is bounded).

We will need the following result concerning the composition semigroup \(\mathcal{S}\) of all self-maps of the interval \([0,1]\) which are fixed on the endpoints.
3.1. Proposition. For every \( f_1, f_2 \in \mathcal{S} \) and \( \varepsilon > 0 \), there exist \( g_1, g_2 \in \mathcal{S} \) such that \( d(f_1 \circ g_1, f_2 \circ g_2) < \varepsilon \).

Proof. For each pair \((m, n)\) of positive integers with \( m \geq n \), let \( P(m, n) \) denote the finite set of piecewise-linear maps \( f \) in \( \mathcal{S} \) satisfying the following conditions:

1) for each \( 0 \leq j \leq m \), \( f(j/m) = k/n \) for some \( 0 \leq k \leq n \); and

2) for each \( 0 \leq j < m \), \( |f((j + 1)/m) - f(j/m)| \leq 1/n \), and \( f \) is linear over the interval \([j/m, (j + 1)/m]\).

Choose \( n \) such that \( 1/n < \varepsilon/4 \), and choose \( m_1, m_2 \) such that \( |f_i(s) - f_i(t)| \leq 1/n \) whenever \( |s - t| \leq 1/m_i \), \( i = 1, 2 \). Then there exist maps \( \phi_i \in P(m_1, n) \) with \( d(f_i, \phi_i) \leq 1/n + 1/2n + 1/2n < \varepsilon/2 \), \( i = 1, 2 \). We show that, for some \( m \geq \max\{m_1, m_2\} \), there exist \( g_1 \in P(m, m_1) \) and \( g_2 \in P(m, m_2) \) with \( \phi_1 \circ g_1 = \phi_2 \circ g_2 \) (note that the compositions are members of \( P(m, n) \)). It then follows that \( d(f_1 \circ g_1, f_2 \circ g_2) < \varepsilon \).

The proof is by induction on \( m_1 + m_2 \). If \( m_1 + m_2 = 2n \) (the least possible value), then \( m_1 = m_2 = n \) and \( \phi_1 = \phi_2 = \text{id} \). In this case take \( m = n \) and \( g_1 = g_2 = \text{id} \).

Now assume \( m_1 + m_2 > 2n \). Suppose first that for some \( j < m_1 \), \( \phi_1(j/m_1) = \phi_1((j + 1)/m_1) \). Then we may consider the corresponding \( \hat{\phi}_1 \in P(m_1 - 1, n) \), obtained topologically by collapsing to a point the arc \([j/m_1, (j + 1)/m_1] \times \phi_1(j/m_1)\) on the graph of \( \phi_1 \). Application of the inductive hypothesis to the pair \( \hat{\phi}_1, \phi_2 \) gives maps \( \gamma_1 \in P(m_0, m_1 - 1) \) and \( \gamma_2 \in P(m_0, m_2) \), for some \( m_0 \geq \max\{m_1 - 1, m_2\} \), such that \( \hat{\phi}_1 \circ \gamma_1 = \phi_2 \circ \gamma_2 \). It's not difficult to see that this implies the corresponding result for the pair \( \phi_1, \phi_2 \). Of
course, the same argument works if $\phi_2(j/m_2) = \phi_2((j + 1)/m_2)$ for some $j < m_2$.

Thus, we may suppose that neither $\phi_i$ is constant on any subinterval. Then there exists a least integer $k$ for which $\phi_i(j/m_i) = k/n$ and $\phi_i((j - 1)/m_i) = \phi_i((j + 1)/m_i) = (k - 1)/n$, for some $1 \leq j < m_i$ and $i = 1, 2$; suppose this holds for $i = 1$. Consider the corresponding $\tilde{\phi}_1 \in P(m_1 - 2, n)$, obtained topologically by identifying the points $((j - 1)/m_1, (k - 1)/n)$ and $((j + 1)/m_1, (k - 1)/n)$ of the restriction $\phi_1/[0, (j - 1)/m_1] \cup [(j + 1)/m_1, 1]$. Applying the inductive hypothesis to the pair $\tilde{\phi}_1, \tilde{\phi}_2$, we obtain maps $\gamma_1 \in P(m_0, m_1 - 2)$ and $\gamma_2 \in P(m_0, m_2)$, for some $m_0 > \max\{m_1 - 2, m_2\}$, such that $\tilde{\phi}_1 \circ \gamma_1 = \tilde{\phi}_2 \circ \gamma_2$. Note that by the choice of $k$, if $\phi_2(i/m_2) = (k - 1)/n$, then either $\phi_2((i - 1)/m_2) = k/n$ or $\phi_2((i + 1)/m_2) = k/n$. Clearly, the above implies the corresponding result for the pair $\phi_1, \phi_2$. This completes the proof of the proposition.

3.2. Remark. If $\sup f_i^{-1}(0) < \inf f_i^{-1}(1)$ for each $i = 1, 2$, then there exists $\delta > 0$ (independent of $\varepsilon$) such that the maps $g_1, g_2$ may be chosen so that $\sup(f_i \circ g_1)^{-1}([0, \delta]) < \inf(f_i \circ g_1)^{-1}([1 - \delta, 1])$, $i = 1, 2$.

3.3. Theorem. Let $X = [0, \infty) \cup K$ be a regular half-line compactification for which there exists a hyperspace retraction $2^X \to C(X)$. Then $X \cong X_n$ for some $n = 0, 1, 2, \ldots$.

Proof. As observed at the beginning of this section, $K$ is either an arc or a simple closed curve. We consider first the case that $K$ is an arc. Suppose $\pi$ is not interior monotone. Then it's not difficult to see that there exists
a proper subarc \( \sigma \) of K, with endpoints \( v \) and \( w \), and points \( t_0, \ldots, t_n \) in \((0, \infty)\), with \( t_0 < t_1 < \cdots < t_n \) and \( n \geq 3 \), such that:

1) \( \pi(t_0) = \pi(t_2) = \cdots = v; \)
2) \( \pi(t_1) = \pi(t_3) = \cdots = w; \)
3) \( \pi([t_0, t_n]) = \sigma \), and \([t_0, t_n]\) is a maximal subinterval in \([0, \infty)\) with respect to this property; and
4) for each \( i = 1, \ldots, n \), the subsets \( \pi^{-1}(v) \cap [t_{i-1}, t_i] \) and \( \pi^{-1}(w) \cap [t_{i-1}, t_i] \) lie in disjoint subintervals.

An application of (3.1) to the maps \( \pi\left|_{[t_0, t_1]} \right. \) and \( \pi\left|_{[t_1, t_2]} \right. \), suitably re-parametrized, shows that for every \( \varepsilon > 0 \) there exist maps \( g_1 : [0, 1] \to [t_0, t_1] \) and \( g_2 : [0, 1] \to [t_1, t_2] \) such that \( g_1(0) = t_1 = g_2(0), g_1(1) = t_0, g_2(1) = t_2, \) and \( d(\pi g_1(t), \pi g_2(t)) < \varepsilon \) for all \( 0 \leq t \leq 1 \). Furthermore, we may assume by (3.2) and the above property 4) that, independently of \( \varepsilon \), there exist neighborhoods \( N(v) \) and \( N(w) \) in \( \sigma \) of \( v \) and \( w \) such that for each \( i = 1, 2, \)

\[
\sup(\pi \circ g_i^{-1}(N(w))) < \inf(\pi \circ g_i^{-1})(N(v)).
\]

For maps \( g_1 \) and \( g_2 \) as above, consider the path \( \alpha : [0, 1] \to 2^X \) between \( \{t_1\} \) and \( \{t_0, t_2\} \), defined by \( \alpha(t) = \{g_1(t), g_2(t)\} \). Let \( R : 2^X \to C(X) \) be a retraction.

If \( \varepsilon > 0 \) is sufficiently small and \( t_0 \) sufficiently large (use the periodicity of \( \pi \)), then for each \( 0 \leq t \leq 1 \), \( \pi R(\alpha(t)) \)

is a small diameter continuum lying in some neighborhood of \( \sigma \) which is a proper subset of K. Since \( \cup R(\alpha(t)): \)

\( 0 \leq t \leq 1 \) is a continuum containing \( R(\alpha(0)) = \{t_1\} \), this implies that \( \cup R(\alpha(t)) \subset (0, \infty) \). Moreover, since

\[
\sup(\pi \circ g_1^{-1}(N(w))) < \inf(\pi \circ g_1^{-1})(N(v)),
\]

we may assume
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\$\varepsilon \text{ sufficiently small and } t_0 \text{ sufficiently large so that}
\U \{R(a(t))\} \subset [0, t_3). \text{ Thus } R(\{t_0, t_2\}) = R(a(1)) \subset [0, t_3).

In fact, we claim that \(R(\{t_0, t_2\}) \subset [0, t_1)\) for all sufficiently large \(t_0\). Otherwise, the small diameter continuum \(R(\{t_0, t_2\})\) would lie in the interval \((t_1, t_3)\), hence
\(R([t, t_0] \cup \{t_2\}) \subset (t_1, t_3)\) for some \(t < t_0\). But by the
maximal nature of \([t_0, t_n]\), \(\pi([t, t_0]) \neq \sigma\), and since
\(R([t, t_0] \cup \{t_2\})\) is arbitrarily close to \(\pi([t, t_0])\) for suf-
ficiently large \(t_0\), this leads to a contradiction.

By another application of (3.1) we obtain maps
\(h_1: [0, 1] \rightarrow [t_0, t_1]\) and \(h_2: [0, 1] \rightarrow [t_2, t_3]\) with
\(h_1(0) = t_0, h_1(1) = t_1, h_2(0) = t_2, h_2(1) = t_3,\) and such
that the maps \(\pi \circ h_1\) and \(\pi \circ h_2\) are arbitrarily close. As
before, we may also assume that \(\sup(\pi \circ h_1)^{-1}(N(v)) < \inf(\pi \circ h_1)^{-1}(N(w)).\) Consideration of the path \(\beta\) in \(2^X\)
between \(\{t_0, t_2\}\) and \(\{t_1, t_3\}\), defined by \(\beta(t) = \{h_1(t),\)
\(h_2(t)\},\) shows that \(R(\{t_1, t_3\}) \subset [0, t_2).\) Continuing in this
fashion we obtain \(R(\{t_n-1, t_n\}) \subset [0, t_{n-1}).\) But an argument
analogous to that given above for \(R(\{t_0, t_2\})\) shows that
\(R(\{t_n-1, t_n\}) \subset (t_{n-1}, \infty).\) This contradiction shows that \(\pi\)
must be interior monotone. Clearly, this implies that
\(X \approx X_0.\)

In the case that \(K\) is a simple closed curve, the same
type of arguments show that the lift \(\tilde{\pi}: [0, \infty) \rightarrow K,\) defined
at the beginning of this section, must be interior monotone.
If \(\tilde{K} = \text{im } \tilde{\pi}\) is unbounded, then in fact \(\tilde{\pi}\) is monotone and
\(X \approx X_1.\) And if \(\tilde{K}\) is bounded, then \(X \approx X_n\) for some \(n > 1.\)
Specifically, \(X \approx X_{2n}\) if the interval \(\tilde{K}\) wraps around \(K\)
exactly n times, while $X \approx X_{2n+1}$ if $K$ wraps around $K$ $n$ times plus a fraction.

4. Conservative Hyperspace Retractions

Recall that a retraction $R: 2^X \rightarrow C(X)$ is conservative if $R(A) \cap A \neq \emptyset$ for each $A \in 2^X$. We show that the topologist's sine curve and the circle with a spiral are the only regular half-line compactifications admitting conservative hyperspace retractions.

4.1. Theorem. Let $X$ be a regular half-line compactification for which there exists a conservative retraction $R: 2^X \rightarrow C(X)$. Then either $X \approx X_0$ or $X \approx X_1$.

Proof. We assume that $X = X(\pi)$, with $\pi = \pi_n$ for some $n > 1$, and show that this leads to a contradiction; the result then follows from (3.3).

Suppose first that $n$ is even. Then for every large integer $k$, $R([k, k+1])$ is a small diameter continuum containing either $k$ or $k+1$, and therefore contained in a small neighborhood in $[0, \infty)$ of either $k$ or $k+1$. If $k$ is sufficiently large, then $\pi R([k - \varepsilon, k + \varepsilon] \cup \{k+1\})$ must be arbitrarily close to $\pi([k - \varepsilon, k + \varepsilon])$, for each $\varepsilon > 0$. Since for all sufficiently small $\varepsilon$, $\pi([k - \varepsilon, k + \varepsilon]) \cap \pi([k + 1 - \varepsilon, k + 1 + \varepsilon]) = \{p\}$, where $p = (1,0) \in S$, consideration of an order arc in $2^X$ between the elements $\{k, k+1\}$ and $[k - \varepsilon, k + \varepsilon] \cup \{k+1\}$ shows that $R([k, k+1])$ cannot lie in a small neighborhood of $k + 1$. An analogous argument involving an order arc between $\{k, k+1\}$ and $\{k\} \cup [k + 1 - \varepsilon, k + 1 + \varepsilon]$ shows that
R({k, k + 1}) cannot lie in a small neighborhood of k. Thus n cannot be even.

Now suppose n is odd. For any large integer k, set
\[ k_1 = \inf \{t: t > k \text{ and } \pi(t) = \pi(k)\} \text{ and } k_2 = \sup \{t: t < k + 1 \text{ and } \pi(t) = \pi(k + 1)\}. \]
Clearly, \( k < k_i < k + 1 \) for each \( i = 1, 2 \). Since \( \pi \) is locally 1-1 at each \( k_i \), but not at \( k \) or \( k + 1 \), arguments analogous to those above show that, for sufficiently large \( k \), \( R({k, k_1}) \) must lie in a small neighborhood of \( k_1 \), and \( R({k_2, k + 1}) \) must lie in a small neighborhood of \( k_2 \). Let \( a: [0,1] \to \mathbb{X} \) be the path between \( \{k,k_1\} \) and \( \{k_2, k + 1\} \) defined by \( a(t) = ((1 - t)k + tk_2, (1 - t)k_1 + t(k + 1)) \). Note that for each \( 0 \leq t \leq 1 \), \( \pi(a(t)) \) is a singleton, and therefore \( R(a(t)) \) must lie in a small neighborhood of one of the points of \( a(t) \). But since for each \( t \) the points of \( a(t) \) remain a constant distance apart, this is inconsistent with the noted properties of \( R(a(0)) \) and \( R(a(1)) \). Thus \( n \) cannot be odd, and this completes the proof that \( X \) is homeomorphic to either \( X_0 \) or \( X_1 \).

5. Construction of Hyperspace Retractions

From this point through section 8, \( X = [0,\infty) \cup K \) will denote one of the regular compactifications \( X_n, n \geq 0 \), described in section 1. Thus, \( K \) is either the interval \( I \) or the circle \( S \). Let \( \pi: X \to K \) be the retraction defined by the periodic surjection \( \pi_n: [0,\infty) \to K \). The construction of a retraction \( R: \mathbb{X} \to C(X) \) is based on the two propositions stated next, whose proofs will be given in sections 7 and 8.
5.1. Proposition. There exists a map $G: 2^X \to C(X)$ with the following properties:

i) $G|C(K) = \text{id}$;

ii) either $G(A) \supseteq \pi(A)$ or $G(A) \subseteq [0, \infty)$;

iii) $G(A) \subseteq K$ if $A \cap K \neq \emptyset$;

iv) $G(A) \supseteq K$ if $A \subseteq [0, \infty)$ and $G(A) \supseteq \pi([\inf A, \sup A])$;

and

v) $G(A) \cap (K \cup A) \neq \emptyset$.

Remark. In the cases $n = 0, 1$, the above property v) may be strengthened by requiring that $G(A) \cap A \neq \emptyset$.

For a given subset $N$ of $C(K)$, let $\mathcal{D}$ be the subset of $C(X) \times C(X)$ defined by $\mathcal{D} = \{(M, N): (M \cup K) \cap N \neq \emptyset$, and either $M \supseteq K \supseteq N$ or $M \cap K \neq \emptyset\}$.

5.2. Proposition. For some neighborhood $N \subseteq C(K)$ of $K$, there exists a map $H: \mathcal{D} \times [0, 1] \to C(X)$ satisfying the following conditions, for every $(M, N) \in \mathcal{D}$ and $0 < t < 1$:

i) $H(M, N, 0) = M$ and $H(M, N, 1) = N$;

ii) either $H(M, N, t) \supseteq M$ or $H(M, N, t) \supseteq N$;

iii) $H(M, N, t) \subseteq [r, \infty) \cup K$ if $M \cup N \subseteq [r, \infty) \cup K$; and

iv) $H(M, N, t) \subseteq [r, s]$ if $M \cup N \subseteq [r, s]$ and $\pi([r, s]) \neq K$.

5.3. Theorem. For $X = [0, \infty) \cup K$ as above, there exists a hyperspace retraction $2^X \to C(X)$.

Proof. Let $F: 2^X \setminus 2^K \to C(X) \setminus C(K)$ denote the "smallest continuum" retraction, defined by

$$F(A) = \begin{cases} [\inf A, \sup A] & \text{if } A \subseteq [0, \infty), \\ \{\inf(A \cap [0, \infty)), \infty\} \cup K & \text{if } A \cap K \neq \emptyset. \end{cases}$$
Define a map $\Theta : 2^X \setminus 2^K \to [0,1]$ by the formula

$$\Theta(A) = \min\{(2/\delta) \cdot \inf(A \cap [0,\omega)) \cdot \rho(\pi(A), \pi(F(A))), 1\},$$

where $0 < \delta < 1$ is chosen such that $\{N \in C(K) : \rho(N, K) < \delta\} \subseteq \eta$, the neighborhood of $K$ in $C(K)$ given by (5.2).

Note that $\Theta(M) = 0$ for all $M \in C(X) \setminus C(K)$.

Let $W = \{A \in 2^X \setminus 2^K : \text{either } A \subseteq [0,\omega) \text{ or } \rho(\pi(A), K) < \delta\}$. Note that $W$ is an open subset of $2^X$, and $C(X) \setminus C(K) \subseteq W$. Let $G : 2^X \to C(X)$ and $H : D \times [0,1] \to C(X)$ be the maps given by (5.1) and (5.2). The desired retraction $R : 2^X \to C(X)$ is defined by

$$R(A) = \begin{cases} H(F(A), G(A), \Theta(A)) & \text{if } A \in W, \\ G(A) & \text{if } A \in 2^X \setminus W. \end{cases}$$

We first verify that for each $A \in W$, $(F(A), G(A)) \in D$, so that $R$ is well-defined. There are two cases to be considered:

1) Suppose $A \in 2^X \setminus 2^K$ with $A \cap K \neq \emptyset$ and $\rho(\pi(A), K) < \delta$. Then $F(A) \supseteq K \supset G(A) \supset \pi(A)$, therefore $\rho(G(A), K) < \delta$ and $G(A) \in H$. Thus $(F(A), G(A)) \in D$.

2) Suppose $A \subseteq [0,\omega)$. Then $F(A) \subseteq [0,\omega)$, and $(F(A) \cup K) \cap G(A) \supset (A \cup K) \cap G(A) \neq \emptyset$, so again $(F(A), G(A)) \in D$.

We next verify that $R/C(X) = \text{id}$. Since $R/C(K) = G/C(K) = \text{id}$, we need only consider $M \in C(X) \setminus C(K)$. Then $\Theta(M) = 0$ and $M \in W$, so $R(M) = H(F(M), G(M), 0) = F(M) = M$.

It remains to show that $R$ is continuous. Since $W$ is open in $2^X$, we have only to verify continuity of $R$ at each $A \in \text{bd } W$. Suppose to the contrary that $R$ is not continuous at some such $A$. Then there exists a sequence $\{A_i\}$ in $W$
converging to $A$, with no subsequence of $\{R(A_i)\}$ converging to $R(A) = G(A)$. In particular, $\emptyset(A_i) \neq 1$ for almost all $i$. There are two cases to be considered.

1) Suppose $A \in 2^K$. Then $\inf(A_i \cap [0,\infty)) \to \infty$, which together with $\emptyset(A_i) \neq 1$ implies that $\rho(\pi(A_i), \pi(F(A_i))) \to 0$. Thus $F(A_i) \to A \in C(K)$, and $G(A_i) \to G(A) = A$. If $A = K$, then $R(A_i) = H(F(A_i), G(A_i), \emptyset(A_i)) \to K$ by the properties ii) and iii) of $H$, contrary to our choice of $\{A_i\}$. Thus $A \notin C(K) \setminus \{K\}$, and $A_i \subset [0,\infty)$ for almost all $i$ since $F(A_i) \to A$.

If $G(A_i) \cap K \neq \emptyset$ for infinitely many $i$, then $G(A_i) \supset \pi(A_i)$ by the property ii) of $G$, and since $F(A_i) \to A \neq K$ and $G(A_i) \to A$, it follows that $G(A_i) \supset \pi(F(A_i))$ for infinitely many $i$. By the property iv) of $G$, $G(A_i) \supset K$, contradicting the convergence of $\{G(A_i)\}$ to $A$.

On the other hand, if $G(A_i) \subset [0,\infty)$ for almost all $i$, then $F(A_i) \cap G(A_i) \supset A_i \cap G(A_i) \neq \emptyset$ by the property v) of $G$, so for almost all $i$, $F(A_i) \cup G(A_i) = [r_i, s_i]$, a subarc of $[0,\infty)$. Since both $\{F(A_i)\}$ and $\{G(A_i)\}$ converge to $A \neq K$, $\pi([r_i, s_i]) \neq K$ for almost all $i$. Then the properties ii) and iv) of $H$ imply that $R(A_i) \to A = R(A)$, again contrary to our choice of $\{A_i\}$.

2) Suppose $A \in 2^X \setminus 2^K$, with $A \cap K \neq \emptyset$ and $\rho(\pi(A), K) \geq \delta$. Then for almost all $i$, $\pi(F(A_i)) = K$ and $\rho(\pi(A_i), K) \geq \delta/2$, yielding $\emptyset(A_i) = 1$, which is impossible. This completes the verification of continuity for $R$.

Finally, we note that the retraction $R$ is conservative if $G$ is, since for each $A \in 2^X$, either $R(A) \supset F(A) \supset A$ or $R(A) \supset G(A)$. Thus, in the cases $n = 0,1$ where a conservative
map $G$ may be chosen, we obtain a conservative hyperspace retraction.

6. Admissible Expansions in $K$

As in the previous section, $X = [0,\infty) \cup K = X_n$ for some $n \geq 0$, with $\pi: X \to K$ the retraction defined by $\pi_n$.

We call a map $e: K \times [0,\infty) \to C(K)$ an expansion if it satisfies the following conditions (for $A \in 2^X$, $e(A,t) = \bigcup \{e(a,t): a \in A\}$):

1) $e(x,t) = e(x,0) = \{x\}$ for all $x$ and $t$;

2) for every $0 \leq s < t$, there exists $\delta > 0$ such that $e(e(x,s),\delta) \subseteq e(x,t)$ for all $x$;

3) for every $A \in 2^K$ and $\delta > 0$, $e(B,\delta) \supseteq A$ for all $B \in 2^K$ sufficiently close to $A$; and

4) for every $A \in 2^K$, $e(A,t) \in C(K)$ for some $t$.

An expansion $e$ is admissible if it permits an extension to a map $\bar{e}: X \times [0,\infty) \to C(X)$ satisfying the above condition 1) and such that, for all $x \in [1,\infty)$ and all $t$, $\bar{e}(x,t) \subseteq [x-1, x+1]$ and $\pi(\bar{e}(x,t)) = e(\pi(x),t)$. We refer to $\bar{e}$ as a "lift" for $e$.

6.1. Lemma. There exists an admissible expansion $e: K \times [0,\infty) \to C(K)$.

Proof. With $d$ the arc-length metric on $K$, we may obtain an expansion by simply setting $e(x,t) = \{y \in K: d(x,y) \leq t\}$. However, this "free" expansion is admissible only if $\pi/(0,\infty)$ is an open map, i.e., only for $n = 0,1$. Thus, for these cases the lemma is trivial, but for $n > 1$, some type of "partial" expansion is required.
Suppose then that $K = S$ and $n > 1$. Let $\omega : (-\infty, \infty) \to S$ be the covering projection defined by $\omega(r) = e^{2\pi i r}$, and let $\tilde{n} : [0, \infty) \to (-\infty, \infty)$ be a lift of the periodic surjection $\pi_n : [0, \infty) \to S$. Then $J = \text{im} \tilde{n}$ is a compact subinterval with length $n/2 > 1$. Let $p, q \in J$ be the points for which $J = [p - 1, q + 1]$. For each $z \in S$, let $z_p, z_q \in (0, 1]$ be the unique values for which $\omega(p - z_p) = z = \omega(q + z_q)$.

Define maps $e_p, e_q : S \times [0, \infty) \to C(S)$ by the formulas

$$
\begin{align*}
e_p(z, t) &= \omega([p - (1 + t)z_p, p - z_p] \cap J), \\
e_q(z, t) &= \omega([q + z_q, q + (1 + t)z_q] \cap J).
\end{align*}
$$

Although the total image function $z \to e_p(z \times [0, \infty))$ is discontinuous at $z = \omega(p)$, the function $e_p$ is continuous; similarly for $e_q$. These maps may be viewed quite simply. For $z \in S$, the restriction $e_p|z \times [0, \infty)$ is clockwise expansion around $S$ from $z$ to $\omega(p)$, where $\omega(p) = \pi([0, 2, 4, \cdots]) = (1, 0)$ is the $\pi$-projection of those "turning points" in $[0, \infty)$ where the direction of travel (towards $\infty$) changes from clockwise rotation about $S$ to counterclockwise rotation. Similarly, $e_q|z \times [0, \infty)$ is counterclockwise expansion from $z$ to $\omega(q)$, where $\omega(q) = \pi([1, 3, 5, \cdots])$ is the $\pi$-projection of those turning points where the direction of travel changes from counterclockwise to clockwise. For even $n$, $\omega(q) = (1, 0)$, while for odd $n$, $\omega(q) = (-1, 0)$.

We show that the map $e : S \times [0, \infty) \to C(S)$, defined by $e(z, t) = e_p(z, t) \cup e_q(z, t)$, is an admissible expansion. The admissibility of $e$ should already be evident from the above discussion of the maps $e_p$ and $e_q$. It remains to verify the expansion conditions 1) through 4).
Condition 1) is obvious. Condition 2) is satisfied with $\delta = t - s/(1 + s)$, since then $(1 + s)(1 + \delta) = (1 + t)$.

The verification of condition 3) is more involved. The basic observation is that, for all $y, z \in S$ and $\delta > 0$,

\[
\begin{cases}
    z_p/(1 + \delta) < y_p < z_p \text{ implies } z \in e_p(y, \delta); \\
    z_q/(1 + \delta) < y_q < z_q \text{ implies } z \in e_q(y, \delta).
\end{cases}
\]

Let $d$ be the metric on $S$ defined by

\[
d(y, z) = \min\{|u - v| : u, v \in (-\infty, \infty) \text{ with } \omega(u) = y \text{ and } \omega(v) = z\}.
\]

The above observation i) implies that for all $y, z$,

ii) if $d(y, z) \leq \min\{z_p, z_q\} \cdot \delta/(1 + \delta)$, then $z \in e(y, \delta)$.

Let $m = \min\{(\omega(p))_q, (\omega(q))_p\}$. Then i) also implies that for all $y$,

iii) if $y_q \leq m\delta/(1 + \delta)$, then $e_p(y, \delta) \supseteq \omega([q, q + y_q])$;

iv) if $y_p \leq m\delta/(1 + \delta)$, then $e_q(y, \delta) \supseteq \omega([p - y_p, p])$.

Assuming $\delta < 1$, iii) implies that for all $y, z$,

\[
\begin{cases}
    \text{if } d(y, z) \leq z_q/2 \leq m\delta/6, \text{ then } e_p(y, \delta) \supseteq \omega([q, q + z_q/2]); \\
    \text{if } d(y, z) \leq z_p/2 \leq m\delta/6, \text{ then } e_q(y, \delta) \supseteq \omega([p - z_p/2, p]).
\end{cases}
\]

We can now verify condition 3). Given $A \in 2^S$ and $\delta > 0$, set $A_p = x_p/2$, for some $x \in A$ such that either $x_p \leq m\delta/3$ or $x_p = \min\{a : a \in A\}$; set $A_q = y_q/2$, for some $y \in A$ such that either $y_q \leq m\delta/3$ or $y_q = \min\{a : a \in A\}$. Let $\eta = \min\{A_p, A_q\} \cdot \delta/(1 + \delta)$. We claim that for every $B \in 2^S$ with $\rho(A, B) < \eta$, $e(B, \delta) \supseteq A$. There are three cases to be considered:

a) Consider $z \in A$ with $z_p \leq A_p$. Then $A_p = x_p/2 \leq m\delta/6$ for some $x \in A$. Choose $y \in B$ with $d(y, x) < \eta < A_p = x_p/2$. 


By iv), $e_q(y,\delta) \supset \omega([p - z_p/2, p])$. Since $z_p \leq x_p/2$, we have $z = \omega(p - z_n) \in \omega([p - x_p/2, p])$. Thus $z \in e_q(y,\delta) \subset e(B,\delta)$.

b) An analogous argument shows that for $z \in A$ with $z_q \leq A_q$, $z \in e(B,\delta)$.

c) Consider $z \in A$ with $z_p > A_p$ and $z_q > A_q$. Choose $y \in B$ with $d(y,z) < \eta \leq \min\{z_p, z_q\} \cdot \delta/(1 + \delta)$. By (ii), $z \in e(y,\delta) \subset e(B,\delta)$.

We next verify condition 4). Note that for each $z \in S$, and sufficiently large $t$, $e_p(z,t) \supset \omega([p - 1, p - z_p])$, the arc (possibly degenerate) traversed in the clockwise direction from $z$ to $w(p)$. Similarly, for large $t$, $e_q(z,t) \supset \omega([q + z_q, q + 1])$, the arc traversed in the counterclockwise direction from $z$ to $w(q)$. If $w(p) = w(q)$, then for every $A \in 2^S$ with $A \not\subset \{w(p)\}$, $e(A,t) = S$ for large $t$. If $w(p) \neq w(q)$, let $\alpha \subset S$ be the subarc traversed in the clockwise direction from $w(q)$ to $w(p)$. Then for each $A \in 2^S$ with $A \not\subset \emptyset$, $e(A,t) = S$ for large $t$, and for $A \subset \alpha$, $e(A,t) = \alpha$ for large $t$. This completes the verification that $e$ is an expansion. And as remarked earlier, $e$ is by its construction admissible.

The above lemma will be used in section 8 for the construction of a map $H$ with the properties specified in (5.2). At present, we apply (6.1) in the case $n > 1$ to obtain a result which will be essential for the construction in the next section of a map $G$ with the properties specified in (5.1).
6.2. Lemma. Let $\pi = \pi_n : [0,\infty) \to S$, $n > 1$. Then there exists a retraction $E : 2^S + C(S)$ with the following properties:

i) $E(A) \supseteq A$ for each $A \in 2^S$; and

ii) for each $A \in 2^S$ and subinterval $L \subseteq [0,\infty)$ such that $A \subseteq \pi(L) \subseteq E(A)$, there exists a subinterval $M \subseteq [0,\infty)$ with $L \subseteq M$ and $\pi(M) = E(A)$.

Proof. Let $e : S \times [0,\infty) \to C(S)$ be an admissible expansion given by (6.1). For each $A \in 2^S$, let $\tau(A)$ denote the smallest value of $t$ for which $e(A,t) \in C(S)$, and define $E : 2^S + C(S)$ by setting $E(A) = e(A,\tau(A))$. Then $E|C(S) = id$, and $E(A) \supseteq A$.

We establish continuity for $E$ by verifying continuity for the function $\tau : 2^S + [0,\infty)$. The lower semi-continuity of $\tau$ is automatic, since $C(S)$ is closed in $2^S$ and $e$ is continuous. Using the expansion properties 2) and 3) of $e$, we show that $\tau$ is upper semi-continuous. Given $A \in 2^S$ and $\varepsilon > 0$, there exists by property 2) a number $\delta > 0$ such that $e(e(B,\tau(A)),\delta) \subseteq e(B,\tau(A) + \varepsilon)$ for all $B \in 2^S$. By continuity of $e$ and property 3), there exists a neighborhood $U$ of $A$ in $2^S$ such that $e(e(B,\tau(A)),\delta) \supseteq e(A,\tau(A))$ for every $B \in U$.

Thus, $e(B,\tau(A) + \varepsilon) \supseteq e(A,\tau(A))$. Also, by application of property 3) to each $\{a\}$, $a \in A$, we may assume the neighborhood $U$ is small enough that for each $B \in U$ and $b \in B$, $e(b,\tau(A) + \varepsilon)$ meets $A$. Thus, each component of $e(B,\tau(A) + \varepsilon)$ meets $A$, and since $A \subseteq e(A,\tau(A)) \subseteq e(B,\tau(A) + \varepsilon)$ and $e(A,\tau(A)) \in C(S)$, it follows that $e(B,\tau(A) + \varepsilon) \subseteq C(S)$.

Then $\tau(B) \leq \tau(A) + \varepsilon$ for every $B \in U$, and $\tau$ is upper semi-continuous.
It remains to verify the property ii). Given \( A \in 2^S \) and a subinterval \( L \subset [0,\infty) \) such that \( A \subset \pi(L) \subset E(A) \), we may assume that \( E(A) \neq S \). Let \( M \supset L \) be a maximal subinterval of \( [0,\infty) \) for which \( \pi(M) \subset E(A) \). We show that \( \pi(M) = E(A) \).

Let \( \tilde{e} : X \times [0,\infty) \to C(X) \) be a lift for \( e \). Since \( A \subset \pi(L) \subset \pi(M) \), we may choose for each \( a \in A \) an element \( \tilde{a} \in M \) with \( \pi(\tilde{a}) = a \). Set \( N_a = \tilde{e}(\tilde{a}, \tau(A)) \). Then \( N_a \) is a subinterval of \( [0,\infty) \) containing \( \tilde{a} \), and \( \pi(N_a) = \pi(\tilde{e}(\tilde{a}, \tau(A))) = e(a, \tau(A)) \subset e(A, \tau(A)) = E(A) \). Since \( \tilde{a} \in M \cap N_a \), \( M \cup N_a \) is a subinterval, with \( \pi(M \cup N_a) \subset E(A) \). By the maximal character of \( M \), we must have \( N_a \subset M \). Thus \( E(A) = \bigcup \{ e(a, \tau(A)) : a \in A \} = \bigcup \{ \pi(N_a) : a \in A \} \subset \pi(M) \), and \( \pi(M) = E(A) \).

7. Construction of the Map \( G \)

We consider first the case \( n > 1 \). Thus, \( K = S \) and \( \pi = \pi_n : [0,\infty) \to S \). As in the proof of (6.1), let \( \omega : (-\infty,\infty) \to S \) be the covering projection defined by \( \omega(r) = e^{2\pi ir} \), and let \( \tilde{\pi} : [0,\infty) \to (-\infty,\infty) \) be a lift of \( \pi \).

The desired map \( G : 2^X \to C(X) \) will be obtained as an extension of the retraction \( E : 2^S \to C(S) \) given by (6.2).

Let \( \mathcal{U} \subset 2^X \) be the collection of those \( A \in 2^X \) which satisfy the following conditions:

a) \( A \subset [0,\infty) \);

b) \( E(\pi(A)) \neq S \); and

c) \( E(\pi(A)) \supset \omega([\inf \tilde{\pi}(A), \sup \tilde{\pi}(A)]) \).

Although condition c) by itself defines a closed subspace of \( 2^X \), \( \mathcal{U} \) is an open subspace. This can be seen from the fact that, since \( E(\pi(A)) \supset \pi(A) = \omega(\tilde{\pi}(A)) \supset \{ \omega(\inf \tilde{\pi}(A)), \omega(\sup \tilde{\pi}(A)) \} \) for each \( A \in 2^X \), \( A \) satisfies conditions b) and
c) if and only if $E(\pi(A)) \cup \omega(\{\inf \pi(A), \sup \pi(A)\}) \neq S$.

Thus conditions b) and c) together define an open subspace of $\mathbb{R}^X$, as does condition a), and therefore $U$ is open.

We claim that for each $A \in U$ and $x \in A$, the continuum $E(\pi(A)) \subset S$ can be "lifted" through $x$, i.e., there exists a continuum $M \subset [0,\infty)$ with $x \in M$ and $\pi(M) = E(\pi(A))$.

Suppose $x \in [i,i+1]$, for some integer $i$; let $L \subset [i,i+1]$ be the subinterval such that $\pi(L) = [\inf \pi(A), \sup \pi(A)]$ (note that $\pi|_{[i,i+1]}$ is a homeomorphism onto $\text{im} \ \pi$). Then $x \in L$, and $\pi(A) \subset \pi(L) = \omega(\pi(L)) \subset E(\pi(A))$ since $A \in U$. The property ii) of the retraction $E$ shows that $L$ may be expanded to an interval $M \subset [i,i+1]$ such that $\pi(M) = E(\pi(A))$.

In particular, if $A \in U$ and $a = \sup A$ is the point of $A$ nearest $S$, with $a \in [i,i+1]$, then there exists a unique interval $M_i \subset [i,i+1]$ with $a \in M_i$ and $\pi(M_i) = E(\pi(A))$. This permits the construction of a map $L: U \to C(X)$ such that for each $A \in U$, $L(A)$ is an "approximate lift" of $E(\pi(A))$ through the point $a = \sup A$. We may construct $L$ according to the following rules:

1) $L(A) = M_i$ if $\min\{a - i, i + 1 - a\} \geq 1/a$;

2) $L(A) = [i, \max M_i]$ if $a - i = 1/2a$, and $L(A) = [\min M_i, i + 1]$ if $i + 1 - a = 1/2a$;

3) $L(A) = M_{i-1} \cup M_i$ if $a = i > 0$, and $L(A) = M_i \cup M_{i+1}$ if $a = i + 1$.

For $1/2a < a - i < 1/a$ or $1/2a < i + 1 - a < 1/a$, $L(A)$ is defined so that $M_i \subset L(A) \subset [i, \max M_i]$ or $M_i \subset L(A) \subset [\min M_i, i + 1]$, respectively, and for $0 < a - i < 1/2a$ or
0 < i + 1 - a < 1/2a, [i, \max M_i] \subset L(A) \subset [\min M_i, \max M_i]

The key properties of the map \( L \) are that \( \sup A \in L(A) \subset [0, \infty) \) and \( \pi(L(A)) = E(\pi(A)) \) for each \( A \in \mathcal{U} \), with \( \inf L(A) \to \infty \) and \( \rho(\pi(L(A)), E(\pi(A))) \to 0 \) as \( \sup A \to \infty \).

The desired map \( G : 2^X 

The desired map \( G : 2^X \to C(X) \) is defined over \( \mathcal{U} \) by modifying \( L \) as follows:

4) \( G(A) = L(A) \) if \( \rho(E(\pi(A)), S) \geq 1/\sup A \);
5) \( G(A) = (\inf L(A), \infty) \cup S \) if \( \rho(E(\pi(A)), S) = 1/(2 \sup A) \);
6) \( G(A) = S \) if \( \rho(E(\pi(A)), S) \leq 1/(4 \sup A) \).

For \( 1/(2 \sup A) < \rho(E(\pi(A)), S) < 1/\sup A \), \( G(A) \) is defined so that \( L(A) \subset G(A) \subset (\inf L(A), \infty) \), and for \( 1/(4 \sup A) < \rho(E(\pi(A)), S) < 1/(2 \sup A) \), \( S \subset G(A) \subset (\inf L(A), \infty) \cup S \).

Note that for \( A \in \mathcal{U} \), either \( G(A) \cap S = \emptyset \) or \( G(A) \supset S \), and \( G(A) \cap (A \cup S) \neq \emptyset \).

Finally, \( G \) is defined over \( 2^X \setminus \mathcal{U} \) by the formula \( G(A) = E(\pi(A)) \). Since \( \mathcal{U} \) is open, it suffices to verify continuity of \( G \) at each \( B \in \text{bd}\mathcal{U} \). Note that, since the condition c) in the definition of \( \mathcal{U} \) is automatically satisfied by each \( B \in \text{bd}\mathcal{U} \), we must have either \( E(\pi(B)) = S \) or \( B \cap S \neq \emptyset \), otherwise \( B \in \mathcal{U} \). If \( G(B) = E(\pi(B)) = S \), then for any \( A \in \mathcal{U} \) near \( B \), either \( G(A) = S \) by virtue of rule 6) above, or \( 1/(4 \sup A) < \rho(E(\pi(A)), S) \), in which case both \( L(A) \) and \( G(A) \) are near \( S \). If \( E(\pi(B)) \neq S \) and \( B \cap S \neq \emptyset \), then for any \( A \in \mathcal{U} \) near \( B \), \( L(A) \) is near \( E(\pi(B)) \) and \( 1/\sup A \leq \rho(E(\pi(A)), S) \), hence \( G(A) = L(A) \) is near \( G(B) = E(\pi(B)) \). Thus \( G \) is a map.

We next verify that \( G \) has the required properties i) through v) of (5.1). Since \( G \) extends \( E \), property i) is clear. Since either \( G(A) \cap S = \emptyset \), \( G(A) \supset S \), or
G(A) = E(π(A)) ⊃ π(A), property ii) is satisfied. Property iii) is immediate from the definition of G over 2^X \setminus U. Property iv) is clear if A ∈ U. On the other hand, if A ⊂ [0,∞) with A ∉ U and G(A) = E(π(A)) ≠ S, then E(π(A)) ∉ ω(\inf π(A), \sup π(A)). However, this contradicts the hypothesis that G(A) ⊃ π(\inf A, \sup A) = ω(\inf π(A), \sup π(A)), since π(\inf A, \sup A) ⊃ [\inf \pi(A), \sup \pi(A)]. Finally, property v) has been previously noted for A ∈ U, and is obvious for A ∈ 2^X \setminus U. This completes the proof of (5.1) in the case n > 1.

In the cases n = 0,1, a streamlined version of the above construction yields a conservative map G: 2^X → C(X) with the required properties. For either K = I or K = S, let E: 2^K → C(K) be any retraction such that E(A) ⊃ A for each A ∈ 2^K. Let \mathcal{U} = \{A ∈ 2^X: A ⊂ [0,∞)\}. As above, an approximate lifting map L: \mathcal{U} → C(X) may be constructed such that for each A ∈ \mathcal{U}, sup A ∈ L(A) ⊂ [0,∞) and π(L(A)) ⊃ E(π(A)), with inf L(A) → ∞ and ρ(π(L(A))), E(π(A))) → 0 as sup A → ∞. In fact, for n = 0, L is constructed in the same manner as above for n > 1. For n = 1, L is constructed such that L(A) ⊂ [0,∞) is the unique lift of E(π(A)) through a = sup A if ρ(E(π(A)),S) ≥ 1/a; a ∈ L(A) ⊂ [a - 2, a + 2] with π(L(A)) ⊃ E(π(A)) if 0 < ρ(E(π(A)),S) < 1/a; and L(A) = [a - 2, a + 2] if E(π(A)) = S.

In either case, L extends to a map G: 2^X → C(X) by the formula G(A) = E(π(A)) for A ∈ 2^X \setminus \mathcal{U}. Properties i) and iii) are immediate from the definition of G. Property ii) is a
consequence of the fact that \( E(\pi(A)) \supseteq \pi(A) \), and that \( G(A) \subset [0,\infty) \) when \( A \subset [0,\infty) \). Property iv) is satisfied vacuously. And finally, \( G(A) \cap A \neq \emptyset \) for all \( A \in 2^X \), since \( G(A) = E(\pi(A)) \supseteq \pi(A) \) if \( A \cap K \neq \emptyset \), and \( G(A) = L(A) \cap \text{sup} A \) if \( A \cap K = \emptyset \).

8. Construction of the Map \( H \)

Let \( e: K \times [0,\infty) \to C(K) \) be an admissible expansion given by (6.1). Set \( N = \{ N \in C(K): e(N,t) = K \text{ for some } t \} \). By the expansion property 3), \( N \) is a neighborhood of \( K \).

The domain \( D \subset C(X) \times C(X) \) of \( H \) can be partitioned into four subdomains as follows:

\[
\begin{align*}
D_1 &= \{ (M,N): M \not\supseteq K \supseteq N \in N \}; \\
D_2 &= \{ (M,N): M \cap K = \emptyset \text{ and } N \subset K \}; \\
D_3 &= \{ (M,N): M \cap K = \emptyset \text{ and } N \not\supseteq K \}; \text{ and} \\
D_4 &= \{ (M,N): M \cap K = \emptyset = N \cap K \text{ and } M \cap N \not= \emptyset \}.
\end{align*}
\]

We will define \( H \) separately over each \( D_i \times [0,1] \).

For \( (M,N) \in D_1 \), set

\[
\begin{align*}
H(M,N,t) &= M, \quad 0 \leq t < 1/4; \\
H(M,N,t) &= K, \quad 1/2 \leq t < 3/4; \text{ and} \\
H(M,N,1) &= N.
\end{align*}
\]

Use the natural path in \( C(X) \) from \( M \) to \( K \) to define \( H(M,N,t) \) for \( 1/4 \leq t \leq 1/2 \), and reverse the \( e \)-expansion \( \{ e(N,t): 0 \leq t < \infty \} \) of \( N \) to \( K \) to define \( H(M,N,t) \) for \( 3/4 \leq t \leq 1 \).

For \( (M,N) \in D_2 \), let \( N^* = e(N, \text{sup} M) \); then \( N \subset N^* \subset C(K) \). Set
\[ \begin{align*}
H(M,N,0) &= M; \\
H(M,N,1/4) &= [\inf M, \infty) \cup K; \\
H(M,N,1/2) &= K; \\
H(M,N,3/4) &= N^*; \text{ and} \\
H(M,N,1) &= N.
\end{align*} \]

Use the natural paths in \( C(X) \) to define \( H(M,N,t) \) for 
\[ 0 \leq t < 1/4 \text{ and } 1/4 < t < 1/2; \text{ reverse the free expansion} \]
(via an arc-length metric) in \( C(K) \) from \( N^* \) to \( K \) to define 
\( H(M,N,t) \) for \( 1/2 \leq t < 3/4; \text{ and reverse the e-expansion} \)
from \( N \) to \( N^* \) to define \( H(M,N,t) \) for \( 3/4 \leq t < 1. \)

For \((M,N) \in \bar{D}_3\), set
\[ \begin{align*}
H(M,N,0) &= M; \\
H(M,N,1/4) &= [\inf M, \infty) \cup K; \\
H(M,N,1/2) &= [\max\{\inf M, \inf N\}, \infty) \cup K; \text{ and} \\
H(M,N,t) &= N, 5/8 \leq t \leq 1.
\end{align*} \]

Use the natural paths in \( C(X) \) to define \( H(M,N,t) \) for all
other \( t \).

Define an index map \( \tau: \bar{D}_4 \to [0, \infty) \) by the formula
\[ \tau(M,N) = \max\{\inf N - \inf M - 2, 0\} \cdot \rho(\pi(N),K). \]

For \((M,N) \in \bar{D}_4\), let \( N^* = \tilde{e}(N, \tau(M,N)) \), where \( \tilde{e} \) is a lift for \( e \).
Then \( N^* \in C(X) \), with \( N \subset N^* \subset [\inf N - 1, \sup N + 1] \). Set
\[ \begin{align*}
H(M,N,0) &= M; \\
H(M,N,1/4) &= [\inf M, \max\{\sup M, \sup N^*\}]; \\
H(M,N,1/2) &= [\max\{\inf M, \inf N^*\}, \max\{\sup M, \sup N^*\}]; \\
H(M,N,5/8) &= [\inf N^*, \max\{\sup M, \sup N^*\}]; \\
H(M,N,3/4) &= N^*; \text{ and} \\
H(M,N,1) &= N.
\end{align*} \]
Use the natural paths in $C(X)$ to complete the definition of $H(M,N,t)$ for $0 \leq t < 3/4$, and reverse the $\varepsilon$-expansion from $N$ to $N^*$ to define $H(M,N,t)$ for $3/4 < t < 1$.

We now verify that $H$ is a map. For $i \neq j$, $D_i \cap D_j \neq \emptyset$ only if $(i,j) = (1,2)$, $(1,3)$, $(1,4)$, $(2,3)$, or $(3,4)$. Since each restriction $H/\partial D_i \times [0,1]$ is continuous, it suffices to check continuity of $H$ at boundary points in the above cases. Considering first the case $(i,j) = (1,2)$, let $(M_k,N_k)$ be a sequence in $\partial D_2$ converging to $(M,N) \in D_1$. Then $\sup M_k \to \infty$, and since $N_k \to N \in N$, we have $N_k^* = K$ for almost all $k$ (use continuity of $\varepsilon$, and the expansion properties 2) and 3). It follows that $H(M_k,N_k,t) \to H(M,N,t)$ whenever $t_k \to t$. The cases $(i,j) = (1,3)$ or $(2,3)$ are routine. Consider a sequence $(M_k,N_k)$ in $\partial D_4$ converging to $(M,N) \in D_1$. Then if $N \neq K$, $\tau(M_k,N_k) \to \infty$ and $N_k^* \to K$; if $N = K$, obviously $N_k^* \to K$. This implies that $H(M_k,N_k,t) \to H(M,N,t)$ whenever $t_k \to t$. Finally, consider a sequence $(M_k,N_k)$ in $\partial D_4$ converging to $(M,N) \in D_3$. Then $\pi(N_k) = K$ for almost all $k$, hence $\tau(M_k,N_k) = 0$ and $N_k^* = N_k$, implying that $H(M_k,N_k,t) \to H(M,N,t)$ whenever $t_k \to t$. This completes the verification of continuity for $H: \partial \times [0,1] \to C(X)$.

Clearly, $H$ satisfies the required conditions i) and ii) of (5.2). Conditions iii) and iv) are also clear, except possibly for $(M,N) \in D_4$ with $N^* \neq N$. However, $N^* \neq N$ implies $\tau(M,N) > 0$, which implies that $\inf N > \inf M + 2$. Then $\inf N^* > \inf N - 1 > \inf M$, and condition iii) is satisfied. And, $\text{diam}(M \cup N) > 2$ implies that $\pi(M \cup N) = K$, so condition iv) is satisfied vacuously. This completes the proof of (5.2).

Let $Y$ be a continuum. A map $\lambda: Y \times Y \to Y$ is called a mean if $\lambda(x,y) = \lambda(y,x)$ and $\lambda(y,y) = y$ for all $x, y \in Y$. A map $\lambda: Y \times Y \to C(Y)$ with the same properties is called a pseudo-mean for $Y$ [7].

Every hyperspace $Z^X$ admits a mean: define $\lambda(A,B) = A \cup B$. If there exists a retraction $Z^X \to C(X)$, then $C(X)$ also admits a mean, and $X$ admits a pseudo-mean. Thus we have yet another necessary condition for the existence of a hyperspace retraction. In this section we describe examples from the class of regular half-line compactifications which show that the existence of a pseudo-mean neither implies nor is implied by the subcontinuum approximation property of section 2, and that both conditions together are still not sufficient for the existence of a hyperspace retraction. Recall that a regular compactification $X = [0,\infty) \cup K$ has the subcontinuum approximation property if and only if the remainder $K$ is either an arc or a simple closed curve. We do not know in general which regular compactifications admit pseudo-means.

9.1. Example. Let $\pi: [0,\infty) \to I$ be the periodic surjection defined as follows:

i) $\pi(k) = 0$ if $k$ is an odd integer;

ii) $\pi(k) = 1$ if $k \equiv 2, 4 \pmod{6}$;

iii) $\pi(k) = -1$ if $k \equiv 6 \pmod{6}$; and

iv) $\pi$ is linear over each interval $[k, k + 1]$. 
Then for $X = X(\pi)$, no retraction $2^X \to C(X)$ exists, since $X \neq X_0$; nonetheless, a pseudo-mean may be constructed for $X$, and in fact $C(X)$ admits a mean.

9.2. Example. Let $\pi: [0, \infty) \to I$ be the periodic surjection defined by:

i) $\pi(k) = 0$ if $k$ is odd;

ii) $\pi(k) = 1$ if $k \equiv 2, 4 \pmod{8}$;

iii) $\pi(k) = -1$ if $k \equiv 6, 8 \pmod{8}$; and

iv) $\pi$ is linear over each interval $[k, k+1]$.

Then $X = X(\pi)$ does not admit a pseudo-mean.

Proof. Suppose there exists a pseudo-mean $\lambda: X \times X \to C(X)$. Let $k$ denote an integer of the form $8n + 2$. Then consideration of $\lambda(k - t, k + t)$, for $0 \leq t \leq 1$ and large $n$, shows that either $\lambda(k - 1, k + 1) \approx \{k + 1\}$ or $\lambda(k - 1, k + 1) \approx \{k + 1\}$. Similarly, either $\lambda(k + 1, k + 3) \approx \{k + 1\}$ or $\lambda(k + 1, k + 3) \approx \{k + 3\}$.

If $\lambda(k - 1, k + 1) \approx \{k + 1\}$, then $\lambda(k, k + 2) \approx \{k\}$; if $\lambda(k + 1, k + 3) \approx \{k + 3\}$, then $\lambda(k, k + 2) \approx \{k + 2\}$.

Thus, either $\lambda(k - 1, k + 1) \approx \{k + 1\}$ or $\lambda(k + 1, k + 3) \approx \{k + 1\}$. Letting $n \to \infty$, we see by continuity of $\lambda$ that, for every $s \in I \subset X$ and the point $0 \in I$, either $\lambda(0, s) \subset \{0, 1\}$ or $1 \in \lambda(0, s')$ for some $s'$ between $0$ and $s$. (Suppose that $\lambda(k - 1, k + 1) \approx \{k + 1\}$ for infinitely many $k$ as above. Then for every $r \in [k - 2, k]$, either $\lambda(r, k + 1) \subset [k, k + 2]$ or $\lambda(r', k + 1) \cap \{k, k + 2\} \neq \emptyset$ for some $r'$ between $k - 1$ and $r$. Note that $\pi(k - 2) = -1$, $\pi(k - 1) = \pi(k + 1) = 0$, and $\pi(k) = \pi(k + 2) = 1$.) An analogous argument shows that either $\lambda(k + 3, k + 5) \approx \{k + 5\}$ or
\[ \lambda(k + 5, k + 7) \approx (k + 5), \] which implies that for every \( s \in I \), either \( \lambda(0, s) \in [-1, 0] \) or \(-1 \notin \lambda(0, s') \) for some \( s' \) between 0 and \( s \). Consequently, \( \lambda(0, s) = \{0\} \) for every \( s \in I \). However, this implies that \( \lambda(k - 1, k) \approx (k - 1) \approx \lambda(k - 1, k + 1) \) and also that \( \lambda(k, k + 1) \approx (k + 1) \approx \lambda(k - 1, k + 1) \), a contradiction. Thus \( X \) does not admit a pseudo-mean.

9.3. Example. Let \( T \) be a triod, with branch point \( v \) and endpoints \( e_1, e_2, \) and \( e_3, \) and let \( \pi: [0, \infty) \to T \) be the periodic surjection defined as follows:

i) \( \pi(k) = v \) if \( k \) is odd;

ii) \( \pi(k) = e_1 \) if \( k \equiv 4 \pmod{8} \);

iii) \( \pi(k) = e_2 \) if \( k \equiv 2, 6 \pmod{8} \);

iv) \( \pi(k) = e_3 \) if \( k \equiv 8 \pmod{8} \); and

v) \( \pi \) is linear over each interval \([k, k + 1]\).

Let \( X = X(\pi) \). It can be shown that \( C(X) \) admits a mean.

9.4. Example. For \( T \) as above, let \( \pi: [0, \infty) \to T \) be the periodic surjection defined by:

i) \( \pi(k) = v \) if \( k \) is odd;

ii) \( \pi(k) = e_1 \) if \( k \equiv 2 \pmod{6} \);

iii) \( \pi(k) = e_2 \) if \( k \equiv 4 \pmod{6} \);

iv) \( \pi(k) = e_3 \) if \( k \equiv 6 \pmod{6} \); and

v) \( \pi \) is linear over each interval \([k, k + 1]\).

Then \( X = X(\pi) \) does not admit a pseudo-mean.

Proof. Suppose there exists a pseudo-mean \( \lambda \). Let \( k \) denote an integer of the form \( 6n + 1 \). Consideration of \( \lambda(k, k + t) \) and \( \lambda(k + 2, k + 2 - t) \), for \( 0 \leq t \leq 1 \) and
large $n$, shows that $\lambda$ must have the following property
with respect to $e_1$: for each $x \in [v,e_1]$, either
$\lambda(v,x) \subset [v,e_1]$ or $e_1 \in \lambda(v,x')$ for some $x'$ between $v$ and
$x$. Of course, $\lambda$ has the analogous properties with respect
to $e_2$ and $e_3$.

Now, consideration of $\lambda(k + 1 - t, k + 1 + t)$, for $0 \leq t \leq 1$ and $k = 6n + 1$ as above, shows that for large $n$,
either $\lambda(k, k + 2) \approx \{k\}$ or $\lambda(k, k + 2) \approx \{k + 2\}$. We may
suppose the former (for infinitely many $n$). Then considera-
tion of $\lambda(k, k + 2 + t)$, for $0 < t < 1$, together with the
above property of $\lambda$ with respect to $e_2$, shows that
$\lambda(v,x) = \{v\}$ for each $x \in [v,e_2]$. But this implies that
$\lambda(k + 2, k + 3) \approx \{k + 2\} \approx \lambda(k + 2, k + 4)$ and also that
$\lambda(k + 4, k + 3) \approx \{k + 4\} \approx \lambda(k + 4, k + 2)$, a contradic-
tion. Thus $X$ does not admit a pseudo-mean.

There also exist regular compactifications
$X = [0, \infty) \cup S$ similar to the above examples. Let
$\pi: [0, \infty) + S$ be the periodic surjection defined by
$\pi(t) = e^{i\pi t}$, $0 \leq t \leq 3 \pmod{4}$, and $\pi(t) = e^{-i\pi t}$, $3 \leq t \leq 4$
$\pmod{4}$. Then for $X = \chi(\pi)$, $C(X)$ admits a mean. On the
other hand, there exist periodic surjections $[0, \infty) + S$ for
which the corresponding compactifications do not admit
pseudo-means. An example is the map $\pi$ defined by
$\pi(t) = e^{i2\pi t}$, $0 \leq t \leq 2 \pmod{3}$, and $\pi(t) = e^{-i2\pi t}$,
$2 \leq t \leq 3 \pmod{3}$.

If there exists a conservative retraction $2^X + C(X)$,
then there exists a conservative pseudo-mean $\lambda: X \times X + C(X)$,
i.e., $\lambda(x,y) \cap \{x,y\} \neq \emptyset$ for all $x,y$. It can be shown that
a regular compactification \( X = [0, \infty) \cup K \) admits a conservative pseudo-mean only if \( X \) is homeomorphic to either \( X_0 \) or \( X_1 \). Thus, in the class of regular half-line compactifications, the existence of a conservative pseudo-mean is equivalent to the existence of a conservative hyperspace retraction. It seems unlikely that this would hold in general, but we do not have a counterexample.

References

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