TOPOLOGY PROCEEDINGS

Volume 12, 1987

Pages 39–46

http://topology.auburn.edu/tp/

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Topology Proceedings

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ISSN: 0146-4124

NON-ZERO-DIMENSIONAL TOPOLOGIES BETWEEN ZERO-DIMENSIONAL ONES

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For a topological space $\langle X, \mathcal{T} \rangle$ and an infinite cardinal κ, we denote by $\mathcal{T}(\kappa)$ the G_κ-modification of $\mathcal{T}$, that is, the topology on X with base $\{\cap \mathcal{G}: \mathcal{G} \subset \mathcal{T}$ and $|\mathcal{G}| < \kappa\}$. A space $\langle X, \mathcal{T} \rangle$ is a P_κ-*space* if $\mathcal{T} = \mathcal{T}(\kappa)$.

A space X is *zero-dimensional at the point* $x \in X$ if x has a local base consisting of clopen sets, and X is *zero-dimensional* if it is zero-dimensional at each of its points.

In [Wi,3.3], Williams proved a theorem which, after typographical correction, reads as follows:

Theorem (Williams). *If* $\langle X, \mathcal{T} \rangle$ *is realcompact, if* κ *is not Ulam-measurable, and if* $\mathcal{T}'$ *is a Tychonoff zero-dimensional topology on* X *with* $\mathcal{T}(\kappa) \subset \mathcal{T}' \subset \mathcal{T}(\kappa^+)$, *then* $\langle X, \mathcal{T}' \rangle$ *is real-compact.*

Williams asked whether the hypothesis of zero-dimensionality on $\mathcal{T}'$ could be removed.

In [BPS_1,3.1(3)], the authors generalized Williams' theorem from the setting of realcompactness to that of α-compactness, and at the same time removed the hypothesis of zero-dimensionality on $\mathcal{T}'$. The question remained, however, whether, for $\kappa \geq \omega_1$, there could in fact be a Tychonoff non-zero-dimensional topology between $\mathcal{T}(\kappa)$ and $\mathcal{T}(\kappa^+)$.[1]

[1] Trivially, for $\kappa = \omega$, there is a space $\langle X, \mathcal{T} \rangle$ and a non-zero-dimensional topology $\mathcal{T}'$ on X with $\mathcal{T}(\kappa) \subset \mathcal{T}' \subset \mathcal{T}(\kappa^+)$; simply take $\langle X, \mathcal{T} \rangle$ non-zero-dimensional and let $\mathcal{T}' = \mathcal{T}$. Also, we are grateful to Amer Bešlagič for pointing out that [En,6.2.19] provides an example of a non-zero-dimensional space $\langle X, \mathcal{T}' \rangle$ for which there is a zero-dimensional topology $\mathcal{T}$ on X with $\mathcal{T}(\omega) = \mathcal{T} \subset \mathcal{T}' \subset \mathcal{T}(\omega^+)$.

In this paper we answer this question, for regular κ, in the affirmative.

The authors would like to thank W. G. Fleissner, whose ideas have been incorporated into many of the techniques of this paper. We are also indebted to E. K. van Douwen for calling [He] to our attention.

The *character of a point* $x \in X$, denoted by $\chi(x,X)$, is defined to be $\min\{|\mathcal{B}| + \omega: \mathcal{B}$ is a local base at x in X$\}$, while the *character* of X, denoted by $\chi(X)$, is defined to be $\sup\{\chi(x,X): x \in X\}$.

As we shall see, the kind of space needed for our technique has large open sets (each with cardinality at least 2^{ω}) and relatively small character. This led us to the following definition: The *amplitude of a space* X *at the point* $x \in X$, denoted by amp(x,X), is defined to be $\min\{|U|: U$ is open in X and $x \in U\}$. The *amplitude* of X, denoted by amp(X) is defined to be $\min\{\text{amp}(x,X): x \in X\}$. Note that this is the same cardinal function that Hewitt calls the dispersion character of X and denotes by $\Delta(X)$ [He, Def. 4].

The reason that we need big open sets and relatively small character is so that the space can be partitioned into 2^{ω} pairwise disjoint dense sets. This led us to another definition: The *congestion number* of a space X, denoted by con(X), is defined to be $\sup\{|\mathcal{D}|: \mathcal{D}$ is a family of pairwise disjoint dense subsets of X$\}$. Relations between amp(X) and con(X), as well as bounds on the sizes of these functions, are discussed in [BPS_2], where the following proposition is proved:

Proposition 1. *If* $\mathrm{amp}(X) \geq \omega$ *and if* $\chi(x,X) \leq \mathrm{amp}(x,X)$ *for all* $x \in X$, *then* $\mathrm{con}(X) = \mathrm{amp}(X)$, *and* X *can be partitioned into* $\mathrm{con}(X)$ *many pairwise disjoint dense sets.*

It is clear that any space X can be partitioned into κ many pairwise disjoint dense sets whenever $0 \neq \kappa < \mathrm{con}(X)$.

We remark that in [He, Theorem 42], Hewitt proved that if X is T_1 and if $\chi(X) \leq \mathrm{amp}(X)$, then X can be partitioned into two disjoint dense sets.

We next prove a result which will guarantee that the construction of our example yields a Tychonoff topology. We denote the real numbers by $\mathbb{R}$.

Proposition 2. *Let* $\langle X,\mathcal{J}\rangle$ *be a* T_1 *zero-dimensional space, let* $r\colon X \to \mathbb{R}$, *and let* $\mathcal{J}_1$ *be the smallest topology on* X *which makes* r *continuous. If* $\mathcal{J}'$ *is the topology on* X *with base* $\{U \cap V\colon U \in \mathcal{J}$ *and* $V \in \mathcal{J}_1\}$, *then* $\mathcal{J}'$ *is Tychonoff.*

Proof. Clearly $\mathcal{J}'$ is T_1. Let $x \in X$ and let V be a $\mathcal{J}'$-neighborhood of x. Then there is a $\mathcal{J}$-clopen set B and an $\varepsilon > 0$ such that $x \in W = B \cap r^{-1}(r(x) - \varepsilon, r(x) + \varepsilon) \subset V$. Define $f\colon \mathrm{cl}_{\mathcal{J}'}W \to \mathbb{R}$ and $g\colon X - W \to \mathbb{R}$ by $f(y) = (1/\varepsilon)|r(y) - r(x)| \wedge 1$ for $y \in \mathrm{cl}_{\mathcal{J}'}W$, and $g(y) = 1$ for $y \in X - W$, and note that f and g are continuous with respect to the topologies on their domains induced by $\mathcal{J}'$. To see that x and $X - W$ are $\mathcal{J}'$-completely separated, it therefore suffices to show that f and g are compatible.

Let $y \in (\mathrm{cl}_{\mathcal{J}'}W) - W$. Then clearly $y \in B$, so $|r(y) - r(x)| \geq \varepsilon$, and hence $f(y) = 1 = g(y)$.

Another requirement of our technique is that our example have a clopen base $\mathcal{B}$ with the property that intersections of decreasing countable sequences from $\mathcal{B}$ have nonempty interiors. This led us to define a family $\mathcal{J}$ of subsets of X to be *κ-saturated* in X if, for every decreasing α-sequence $\langle A_\xi : \xi < \alpha \rangle$ in $\mathcal{J}$ with $\alpha < \kappa$, $\mathrm{int}_X \bigcap_{\xi<\alpha} A_\alpha \neq \emptyset$.

If a space X has an ω_1-saturated base, then clearly X is an almost-P-space (i.e., every nonempty G_δ in X has nonempty interior [Le]), but a P_κ-space need not have a κ-saturated, or even an ω_1-saturated, base, as the following example shows: Let κ be a regular cardinal and let X be the full κ-ary tree of height ω; that is, X has a smallest element and each node branches κ times. Let a point $x \in X$ have as its basic neighborhoods sets of the form $\{x\} \cup A$, where A consists of all but $<\kappa$ nodes immediately above x, together with all nodes above these. Then it is easy to see that X is a P_κ-space with no ω_1-saturated base.

In view of [GJ, 6ST] and the following proposition, $\beta\omega - \omega$ is an example of a space with an ω_1-saturated clopen base that is not a P-space (= P_{ω_1}-space).

Proposition 3. *Every countably compact, zero-dimensional, almost-*P*-space* X *has an* ω_1*-saturated clopen base.*

Proof. Let $\mathcal{B}$ be a base for X consisting of nonempty clopen sets. Since X is an almost-P-space, it suffices to note that every countable decreasing sequence from $\mathcal{B}$ has nonempty intersection.

Our next proposition describes the principal construction technique for our example.

Proposition 4. *Let* $\langle X,\mathcal{J}\rangle$ *be a* T_1*-space with an* ω_1*-saturated clopen base, and assume that* $X = \cup_{t\in\mathbb{R}} A_t$, *where the* A_t*'s are pairwise disjoint dense subsets of* X. *Then there is a Tychonoff topology* $\mathcal{J}'$ *on* X *such that* $\mathcal{J} \subset \mathcal{J}'$ *and such that* $\mathcal{J}'$ *is not zero-dimensional at any point of* X.

Proof. Define $r: X \to \mathbb{R}$ by $r(x) = t$ if $x \in A_t$. Let $\mathcal{J}_1$ be the smallest topology on X which makes r continuous and let $\mathcal{J}'$ be the topology on X with base $\{U \cap V: U \in \mathcal{J}$ and $V \in \mathcal{J}_1\}$. Then by Proposition 2, $\mathcal{J}'$ is Tychonoff.

To see that $\mathcal{J}'$ is not zero-dimensional at $x \in X$, we assume the contrary. Suppose $r(x) = a$. Then there is a $\mathcal{J}'$-clopen set U with $x \in U$ and with $r^{-1}(\{a+1\}) \subset X - U$.

Let $\mathcal{B}$ be an ω_1-saturated clopen base for $\mathcal{J}$. We shall show, recursively, that there exist sequences $\langle B_\xi: \xi < \omega_1\rangle$ and $\langle \varepsilon_\xi: \xi < \omega_1\rangle$ such that:

(1) For every $\xi < \omega_1$, $B_\xi \in \mathcal{B}$, $\varepsilon_\xi \in \mathbb{R}$ with $0 < \varepsilon_\xi < 1$, and $B_\xi \cap r^{-1}[a,a+\varepsilon_\xi) \subset U$.

(2) If $\xi < \eta < \omega_1$, then $B_\eta \subset B_\xi$ and $\varepsilon_\xi < \varepsilon_\eta$.

Since $x \in U \in \mathcal{J}'$, there is $B_0 \in \mathcal{B}$ and $\varepsilon_0 \in \mathbb{R}$ with $0 < \varepsilon_0 < 1$ and $B_0 \cap r^{-1}(a-\varepsilon_0,a+\varepsilon_0) \subset U$.

Assume now that $0 < \alpha < \omega_1$ and that $\langle B_\xi: \xi < \alpha\rangle$ and $\langle \varepsilon_\xi: \xi < \alpha\rangle$ have been constructed satisfying (1) and (2).

Case 1. $\alpha = \beta + 1$. Since $A_{a+\varepsilon_\beta}$ is dense in $\langle X,\mathcal{J}\rangle$, there is $y \in B_\beta \cap r^{-1}(\{a+\varepsilon_\beta\})$. Suppose $y \notin U$. Then there is $W \in \mathcal{J}$ and $\delta \in \mathbb{R}$ such that $0 < \delta \leq \varepsilon_\beta$ and $y \in W \cap r^{-1}(a+\varepsilon_\beta-\delta, a+\varepsilon_\beta+\delta) \subset X - U$. Pick $t \in (a+\varepsilon_\beta-\delta,a+\varepsilon_\beta)$. There is

$z \in W \cap B_\beta \cap r^{-1}(\{t\})$, and we have $z \in B_\beta \cap r^{-1}[a,a+\varepsilon_\beta) \subset U$. We also have $z \in W \cap r^{-1}(a+\varepsilon_\beta-\delta, a+\varepsilon_\beta+\delta) \subset X - U$, a contradiction. Thus $y \in U$, and so there are $V \in \mathcal{J}$ and $\sigma \in \mathbb{R}$ with $0 < \sigma < 1-\varepsilon_\beta$ and $y \in V \cap r^{-1}(a+\varepsilon_\beta-\sigma, a+\varepsilon_\beta+\sigma) \subset U$. There is $B_\alpha \in \mathcal{B}$ with $B_\alpha \subset B_\beta \cap V$, and we set $\varepsilon_\alpha = \varepsilon_\beta+\sigma$. Then conditions (1) and (2) are easily verified.

Case 2. α is a limit ordinal. Note that $\langle B_\xi : \xi < \alpha \rangle$ is a decreasing α-sequence in the ω_1-saturated family $\mathcal{B}$, and hence $B = \mathrm{int}_{\mathcal{J}} \bigcap_{\xi<\alpha} B_\xi$ is nonempty. Let $\varepsilon' = \sup\{\varepsilon_\xi : \xi < \alpha\}$. We claim that $\varepsilon' < 1$.

If not, then $\varepsilon' = 1$ and there is $y \in B \cap r^{-1}(\{a+1\})$. Then $y \in X - U$, and so there are $V \in \mathcal{J}$ and $\delta \in \mathbb{R}$ with $0 < \delta \leq 1$ and $y \in V \cap r^{-1}(a+1-\delta, a+1+\delta) \subset X - U$. There is $\beta < \alpha$ such that $\varepsilon_\beta > 1 - \delta$, and there is $t \in (a+1-\delta, a+\varepsilon_\beta)$. There is also $z \in B \cap V \cap r^{-1}(\{t\})$. Then $z \in B_\beta \cap r^{-1}[a, a+\varepsilon_\beta) \subset U$ and also $z \in V \cap r^{-1}(a+1-\delta, a+1+\delta) \subset X - U$, a contradiction. We conclude $\varepsilon' < 1$.

We now pick $B_\alpha \in \mathcal{B}$ with $B_\alpha \subset B$, and we set $\varepsilon_\alpha = \varepsilon'$. Conditions (1) and (2) are again easily verified, and hence the recursion is complete.

Now the existence of the family $\langle (\varepsilon_\xi, \varepsilon_{\xi+1}) : \xi < \omega_1 \rangle$ obviously contradicts the fact that $\mathbb{R}$ has countable cellularity.

We are now ready to describe our example. By the *κ-box product topology* on a Cartesian product $\prod_{i\in I} X_i$, we mean the topology generated by the base $\{\bigcap_{i\in J} pr_i^{-1}(U_i) : U_i$ open in X_i, $J \subset I$, and $|J| < \kappa\}$.

Proposition 5. *Let* κ *be an uncountable regular cardinal, let* $X = 2^{\kappa}$*, and let* $\mathcal{J}$ *be the* κ*-box product topology on* X. *Then there is a Tychonoff topology* $\mathcal{J}'$ *on* X *with* $\mathcal{J}(\kappa) \subset \mathcal{J}' \subset \mathcal{J}(\kappa^{+})$ *such that* $\mathcal{J}'$ *is not zero-dimensional at any point of* X.

Proof. Since κ is regular, $\mathcal{J} = \mathcal{J}(\kappa)$, and it is easy to check that the canonical base for $\mathcal{J}$ is ω_1-saturated and clopen. Since $\chi(X) \leq \kappa^{<\kappa} \leq 2^{\kappa} = \mathrm{amp}(X) \geq 2^{\omega}$, by Proposition 1 we have $X = \cup_{t\in\mathbb{R}} A_t$, where the A_t's are pairwise disjoint and $\mathcal{J}$-dense in X. Then by Proposition 4 there is a Tychonoff topology $\mathcal{J}'$ on X such that $\mathcal{J}'$ is not zero-dimensional at any point of X. Since $\mathcal{J}(\kappa^{+})$ is the discrete topology on X, we then have $\mathcal{J}(\kappa) \subset \mathcal{J}' \subset \mathcal{J}(\kappa^{+})$.

It can be shown, without using Proposition 1, that if $X = \langle 2^{\kappa}, \mathcal{J} \rangle$ where $\mathcal{J}$ is the κ-box product topology on 2^{κ}, then $X = \cup_{t\in\mathbb{R}} A_t$, where the A_t's are pairwise disjoint and dense in X. There is, in fact, an elementary proof of this for any space X for which $2^{\omega} \cdot w(X) \leq \mathrm{amp}(X)$. For the proof, one picks the dense sets recursively, using an obvious diagonal method.

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