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The known proof that an isometry $f: X \to X$ on a compact metric space $(X,d)$ is onto depends on the sequential compactness of $X$ and the sequence $\{y, f(y), f^2(y), \cdots\}$ of iterates of any point $y \in X$. Our proof shows the exact role that the total boundedness and completeness of $X$ play in the result mentioned above. Our techniques can easily be generalized to uniform spaces, with interesting consequences.

This work is complementary to [1]; throughout, we use the terminology of [1] and [4].

For the sake of convenience, let us define an $\xi$-net ($\xi > 0$) for a pseudometric space $(X,d)$ as a finite cover $\mathcal{J}_\xi = \{U_1, \cdots, U_j\}$ of $X$ such that diam $U_i \leq \xi$, for $i = 1, 2, \cdots, j$.

The following lemma is obvious but crucial to the work that follows.

*Lemma* 1. *If the pseudometric space* $(X,d)$ *has an* $\xi$*-net* $\mathcal{J}_\xi$ *then it has a minimum* $\xi$*-net* $\mathcal{J}'_\xi$ *(in the sense that every* $\xi$*-net for* $X$ *will have at least as many elements as* $\mathcal{J}'_\xi$*).*

*Definition* 2. Let $(X,d)$ be a pseudometric space, $f: X \to X$ a function and $\alpha > 0$. The map $f$ is said to be $\alpha$-expansive if $d(f(x), f(y)) \geq d(x,y)$ whenever $d(f(x), f(y)) \leq \alpha$. The map $f$ is said to be expansive if it is $\alpha$-expansive, for all $\alpha > 0$.

Note that isometries are expansive maps but it is possible that a $\mathcal{E}$-isometry (see Definition 4.4 of [1]) is not $\mathcal{E}$-expansive.

A weaker version of the following result is known (see Lemma 3.2 of [3]). Our method of proof plays a role in the work that follows.

*Lemma* 3. *Let* $(X,d)$ *be a totally bounded pseudometric space and* $f\colon X \to X$ *a* $\xi_o$*-expansive map, for some* $\xi_o > 0$. *Then* $f(X)$ *is dense in* $X$.

*Proof*. Suppose $f(X)$ is not dense in $X$. Pick $y \in X - f(X)$ such that $0 < 2\xi \le d(y,f(X))$. Without loss of generality, let us assume that $\xi \le \xi_o$. Let $\mathcal{J}_\xi = \{U_1,\cdots,U_j\}$ be a minimum $\xi$-net for $X$. Then $y \in$ some $U_i$, which implies that $U_i \cap f(X) = \phi$. Then $\mathcal{J}'_\xi = \{f^{-1}(U_k) \mid k \neq i\}$ is also a $\xi$-net for $X$, because $f(X) \subset \cup_{k\neq i} U_k$ which implies that $X \subset \cup_{k\neq i} f^{-1}(U_k)$ with each diam $f^{-1}(U_k) \le \xi$. This contradicts the minimality of $\mathcal{J}_\xi$. Therefore, $f(X)$ is dense in $X$.

*Lemma* 4. *Let* $(X,d)$ *be a complete metric space and* $f\colon X \to X$ *a continuous* $\xi_o$*-expansive map, for some* $\xi_o > 0$, *such that* $f(X)$ *is dense in* $X$. *Then* $f(X) = X$.

*Proof*. It is easily seen that $f(X)$ is a complete subspace of $X$, which implies that $f(X)$ is a closed subspace of $X$; therefore, $f(X) = X$.

*Corollary* 5 (Banach-Ulam generalized). *Let* $(X,d)$ *be a compact metric space and* $f\colon X \to X$ *an isometry or a continuous* $\xi_o$*-expansive map, for some* $\xi_o > 0$. *Then* $f$ *is onto.*

*Proof.* Immediate from Lemmas 3 and 4.

Lemma 3 may lead one to believe that Corollary 5 is valid for a class of totally bounded metric spaces which is significantly larger than the class of compact metric spaces. However, the following result leaves little hope for any major improvement of Corollary 5. Nonetheless, significant improvements are possible. (See Proposition 16 and subsequent questions.)

*Proposition 6. There exists a totally bounded, topologically complete, pathwise connected and locally pathwise connected subspace* $X$ *of the euclidean plane and an isometry* $f: X \to X$ *which is not onto.*

*Proof.* Let $Y$ be the closed unit ball in the euclidean plane centered at the origin (i.e. the closed 2-euclidean ball). Let $g: Y \to Y$ be the rotation isometry defined by $g(\alpha e^{i\theta}) = \alpha e^{i(\theta+\sqrt{2}\pi)}$. Then, letting $a = (1,0)$, let $X = Y - \{g(a), g^2(a), \cdots\}$. Note that $a \in X$, since $a = e^{i2\pi n}$ and $g^k(a) = e^{i\sqrt{2}\pi k}$, $n, k = 1,2,\cdots$. (Consequently, $a \neq g^k(a)$, for $k = 1,2,\cdots$.) Next note that $g^{-1}: X \to X$ (if $g^{-1}(x) = g^k(a)$, then $x = gg^{-1}(x) = g^{k+1}(a)$; therefore, if $x \in X$ we get that $g^{-1}(x) \in X$). Furthermore, $g^{-1}$ is not onto since $a \notin g^{-1}(x)$ (say $a = g^{-1}(x)$; then $x = g(a) \notin X$). Since it is clear that $g^{-1}$ is an isometry and $X$ satisfies all requirements (note that $X$ is topologically complete because it is a $G_\delta$-subspace of the euclidean plane), the proof is complete.

We are now ready to extend a large number of results on $\alpha$-non-expansive and $\alpha$-expansive maps to uniform spaces. We start by expanding Definition 4.4 of [1].

For reasons which will soon be clear, a family $\theta = \{\rho_\lambda\}_{\lambda\in\Lambda}$ of pseudometrics on a set X will be called a *subgage* for a uniformity $\mathcal{U}$ on X if $\{\{(x,y) \in X \times X \mid \rho_\lambda(x,y) < \xi\} \mid \xi > 0 \text{ and } \lambda \in \Lambda\}$ generates $\mathcal{U}$ (i.e. is a subbase for $\mathcal{U}$). $\theta$ will be said to be *separating* if for $x \neq y$ in X there exists $\rho_\lambda \in \theta$ such that $\rho_\lambda(x,y) \neq 0$. We will also call $\theta$ a *full subgage* if $\theta$ is closed with respect to sups of finite sets of pseudometrics (i.e. if $\rho_1,\cdots,\rho_n \in \theta$ then $\sup\{\rho_1,\cdots,\rho_n\} \in \theta$). Clearly, every subgage $\theta$ automatically generates a full subgage $\theta^* = \{\sup\{\rho_1,\cdots,\rho_n\} \mid \{\rho_1,\cdots,\rho_n\} \subset \theta,\ n = 1,2,\cdots\}$.

*Standing Assumption.* Henceforth, all uniform spaces $(X,\mathcal{U})$ will be assumed to be *separated* (if $x \neq y$ in X then there exists $U \in \mathcal{U}$ such that $(x,y) \notin U$) and all subgages will be assumed to be separating. Uniform spaces will automatically carry the corresponding uniform topology. Topological spaces will be assumed to be Hausdorff, unless they are generated by pseudometrics.

The following restatement of Theorem 18 on p. 189 of [4] is more convenient for our work.

*Proposition* 7. *Let* $(X,\mathcal{U})$ *be a uniform space and* $\theta$ *a subgage for* $\mathcal{U}$. *Then*

(a) *The full subgage* $\theta^*$ *generates a base for* $\mathcal{U}$,

(b) $\theta^{**} = \{\rho | \rho$ *is a pseudometric for* X *and, for each* $\xi > 0$, *there exists* $\delta > 0$ *and* $\rho' \in \theta^*$ *such that* $\rho'(x,y) < \delta$ *implies* $\rho(x,y) < \xi\}$ *is the gage for* $\mathcal{U}$.

Henceforth, we will use the notation $\theta$, $\theta^*$, $\theta^{**}$ with the meaning established in Proposition 7.

*Definition* 8. Let $(X,\mathcal{U})$ be a uniform space and $\theta = \{\rho_\lambda\}_{\lambda\in\Lambda}$ be a subgage for $\mathcal{U}$. Given $\xi > 0$, a function $f: X \to X$ is said to be

(a) $\xi$-*expansive with respect to* $\theta$ (or $(\theta,\xi)$-expansive) if $\rho_\lambda(f(x),f(y)) \geq \rho_\lambda(x,y)$, whenever $\rho_\lambda(x,y) < \xi$ and $\lambda \in \Lambda$,

(b) *expansive with respect to* $\theta$ (or $\theta$-expansive) if f is $(\theta,\xi)$-expansive, for all $\xi > 0$,

(c) $(\theta,\xi)$-isometry if $\rho_\lambda(f(x),f(y)) = \rho_\lambda(x,y)$, whenever $\rho_\lambda(x,y) < \xi$ and $\lambda \in \Lambda$,

(d) $\theta$-isometry if f is a $(\theta,\xi)$-isometry, for all $\xi > 0$.

*Definition* 9. Let $(X,\mathcal{U})$ be a uniform space and $\theta = \{\rho_\lambda\}_{\lambda\in\Lambda}$ be a subgage for $\mathcal{U}$. X is said to be

(a) $\theta$-*totally bounded* if $(X,\rho_\lambda)$ is a totally bounded pseudometric space, for each $\lambda \in \Lambda$,

(b) *sub-totally bounded* (totally bounded) if there exists a subgage (gage) $\theta$ for $\mathcal{U}$ such that X is $\theta$-totally bounded,

(c) $\theta$-*complete* if each $(X,\rho_\lambda)$ is a complete pseudometric space,

(d) *sub-complete* if there exists a subgage $\theta$ for $\mathcal{U}$ such that X is $\theta$-complete.

*Lemma* 10. *Let* $(X,\mathcal{U})$ *be a uniform space,* $\theta$ *a subgage for* $\mathcal{U}$ *and* $f\colon X \to X$ *a function. The following are valid:*

(a) $X$ *is* $\theta$*-totally bounded iff* $X$ *is* $\theta^{**}$*-totally bounded,*

(b) $f$ *is* $(\theta,\xi)$*-expansive iff* $f$ *is* $(\theta^{*},\xi)$*-expansive,*

(c) $f$ *is* $\theta$*-expansive iff* $X$ *is* $\theta^{*}$*-expansive.*

*Proof.* Let us first note that the "if" parts of (a), (b) and (c) are trivial.

The "only if" part of (a). First, we show that $X$ is $\theta^{*}$-totally bounded: Let $\rho_1,\rho_2 \in \theta$ and let $\rho = \sup\{\rho_1,\rho_2\}$. Let $\{x_n\}$ be a sequence in $X$ and $\{w_k\}$ be a $\rho_1$-Cauchy subsequence of $\{x_n\}$; then let $\{z_j\}$ be a $\rho_2$-Cauchy subsequence of $\{w_k\}$. It follows easily that $\{z_j\}$ is a $\rho$-Cauchy subsequence of $\{x_n\}$. This shows that $\rho$ is a totally bounded pseudometric for $X$. Since the preceding argument immediately generalizes to $\sup\{\rho_1,\cdots,\rho_n\}$, for any finite $\{\rho_1,\cdots\rho_n\} \subset \theta$, we get that $X$ is $\theta^{*}$-totally bounded. It follows easily from Proposition 7(b) that $X$ is $\theta^{**}$-totally bounded (note that, in Proposition 7(b), a $\delta$-net for $\rho'$ is an $\xi$-net for $\rho$).

The "only if" part of (b) is routine and automatically implies the "only if" part of (c).

Lemma 10 suggests the following questions: Let $(X,\mathcal{U})$ be a uniform space and $\theta$ a subgage for $\mathcal{U}$. If $X$ is $\theta$-complete is $X$ $\theta^{*}$-complete? If $f\colon X \to X$ is $(\theta,\xi)$-expansive, for some $\xi > 0$, is $f$ also $(\theta^{**},\xi)$-expansive? We still do not know the answer to the second question but the referee has kindly outlined a remarkably simple negative solution for the first question, which is reproduced in the following example.

*Example* 11. Let $X = \{\pm \frac{1}{n} | n = 1,2,\cdots\}$ with the uniformity induced by the Euclidean metric on the real line $E^1$. Let $\theta = \{\rho_1,\rho_2\}$, where

$$\rho_1(x,y) = \begin{cases} |x - y| \text{ if } x > 0,\ y > 0, \\ \max\{x,y,0\}, \text{ otherwise,} \end{cases}$$

$$\rho_2(x,y) = \begin{cases} |x - y| \text{ if } x < 0,\ y < 0, \\ \max\{-x,-y,0\}, \text{ otherwise} \end{cases}$$

It is easily seen that $\rho_1$ and $\rho_2$ are pseudometrics on X. It is also easily seen that $(X,\rho_1)$ and $(X,\rho_2)$ are complete pseudometric spaces. It is clear that any $\rho_1$-Cauchy sequence which is not eventually constant will $\rho_1$-converge to any $-\frac{1}{k} \in X$, while any $\rho_2$-Cauchy sequence which is not eventually constant will $\rho_2$-converge to any $\frac{1}{k} \in X$. Letting $\rho = \sup\{\rho_1,\rho_2\}$, one easily sees that

$$\rho(x,y) = \begin{cases} |x - y| & \text{if } xy > 0, \\ \max\{|x|,|y|\} & \text{if } xy < 0. \end{cases}$$

Consequently, $\theta$ generates the uniformity of X, since it is easily seen that $\rho$ does. However, X is not $\theta^*$-complete, because $\{\frac{1}{n}\}$ is a $\rho$-Cauchy sequence which does not $\rho$-converge in X.

It is noteworthy that a uniform space $(X,\mathcal{U})$ may be sub-complete without being complete (see [4]): The space $\Omega_o$ of countable ordinals with the order uniformity is totally bounded and sub-complete but it is not complete, since it is not compact. (The deatils appear at the top of p. 553 of [2], where "gage" should be replaced by "subgage.") However, the following result shows that $\Omega_o$ is indeed sub-complete with respect to the gage for its order uniformity.

The reason that $\Omega_0$ is not compelte is that the net $\{S_\alpha = \alpha\}_{\alpha\in\Omega_0}$ is $\rho$-Cauchy, for each $\rho$ in its gage, and it does converge to many points in $(\Omega_0,\rho)$, but it does not converge in ($\Omega_0$, order topology).

*Proposition* 12. *If a uniform space* $(X,\mathcal{U})$ *is pseudocompact then* X *is totally bounded and sub-complete.*

*Proof.* Let $\theta = \{\rho_\lambda\}_{\lambda\in\Lambda}$ be any subgage for $\mathcal{U}$. Then, the identity map $j\colon (X,\mathcal{U}) \to (X,\rho_\lambda)$ is continuous, for each $\lambda \in \Lambda$. Consequently, each $(X,\rho_\lambda)$ is pseudocompact. Since $\rho_\lambda$ is a pseudometric on X, we then get that each $(X,\rho_\lambda)$ is compact, which proves that each $(X,\rho_\lambda)$ is totally bounded and complete; that is, $(X,\mathcal{U})$ is totally bounded and sub-complete.

Again, as suggested by the referee, the space X of Example 11 can be used to show that the converse of Proposition 12 is false: Using the notation of Example 11, we get that X is $\theta$-complete; hence, X is sub-complete. Also, X is clearly $\theta$-totally bounded, which implies that X is $\theta^{**}$-totally bounded, by Lemma 10(a); hence, X is totally bounded. However, X is not pseudocompact (for example, $f\colon X \to E^1$, defined by $f(\pm \frac{1}{n}) = n$, is an unbounded continuous function).

It is noteworthy that Proposition 12 leads naturally to a class of spaces which contains the class of pseudocompact spaces and the class of totally bounded uniform spaces.

*Definition* 13. A uniform space $(X,\mathcal{U})$ is said to be *uniformly pseudocompact* if every uniformly continuous real-valued function on X is bounded.

*Proposition* 14. *A totally bounded uniform space* $(X,\mathcal{U})$ *is uniformly pseudocompact.*

*Proof.* Let $\theta$ be a subgage for $\mathcal{U}$. Next, let $f: X \to E^1$ be a uniformly continuous function. Then, letting $\rho(x,y) = |f(x) - f(y)|$, for each $x,y \in X$, we get that $\rho$ is a uniformly continuous pseudometric for X; this means that $\rho \in \theta^{**}$, which implies that $\rho$ is bounded, by Lemma 10(a). The boundedness of $\rho$ clearly implies that f is bounded, which completes the proof.

We conclude with further generalizations and improvements of known results, including the Banach Contraction Principle.

*Proposition* 15. *Let* $(X,\mathcal{U})$ *be a uniform space and* $\theta$ *a subgage for* $\mathcal{U}$. *If* X *is* $\theta$*-totally bounded and* $f: X \to X$ *is a* $(\theta,\xi_o)$*-expansive map, for some* $\xi_o > 0$, *then* $f(X)$ *is dense in* X.

*Proof.* By Lemma 10(b), f is a $(\theta^*,\xi_o)$-expansive map. Consequently, for each $\rho \in \theta^*$, $f(X)$ is dense in $(X,\rho)$, by the proof of Lemma 3. Since $\theta^*$ generates a base for $\mathcal{U}$ we then get that $f(X)$ is dense in X.

Recall that a space X is said to be sequential if any sequentially closed subset of X is closed.

*Proposition* 16. *Let* $(X,\mathcal{U})$ *be a sequential, sequentially compact uniform space and* $\theta$ *a subgage for* $\mathcal{U}$. *If* $f\colon X \to X$ *is a continuous* $(\theta,\xi_o)$*-expansive map, for some* $\xi_o > 0$, *then* $f(X) = X$.

*Proof*. By Propositions 12 and 15, $f(X)$ is dense in $X$.

Assume there exists $y \in X - f(X)$ and pick a sequence $\{x_n\}$ in $X$ such that $\lim_n f(x_n) = y$. Let $\{x_{n_k}\}$ be a convergent subsequence of $\{x_n\}$; say $\lim_k x_{n_k} = x$. We will show that $y = f(x)$: Suppose not. Pick $\rho \in \theta$ such that $\rho(f(x),y) > 0$. Since $f\colon (X,\mathcal{U}) \to (X,\rho)$ is continuous, we get that $\{f(x_{n_k})\}$ $\rho$-converges to the distinct points $f(x)$ and $y$ which are a positive $\rho$-distance apart, a contradiction.

Since $y = f(x)$ contradicts the assumption that $y \in X - f(X)$, we have proved that $f(X) = X$.

Note that the preceding result applies to the space of countable ordinals with the order topology.

Naturally, Proposition 16 raises a variety of questions, none of which appears trivial: Is the hypothesis that $(X,\mathcal{U})$ be sequential superfluous? Is the conclusion of Proposition 16 valid for any countably compact (pseudocompact) space $(X,\mathcal{U})$? If $(X,\mathcal{U})$ has a subgage $\theta$ such that $X$ is $\theta$-totally bounded and $\theta$-complete, and $f\colon X \to X$ is a $(\theta,\xi)$-expansive map, for some $\xi > 0$, is $f(X) = X$? (The preceding questions remain open even if $f$ is a $\theta$-isometry.)

*Definition* 17. Let $(X,\mathcal{U})$ be a uniform space, $\theta$ a subgage for $\mathcal{U}$ and $f\colon X \to X$ a function. We say that $f$ is a

$\theta$-*contraction* if there exists $0 \leq \alpha(\rho) < 1$, for each $\rho \in \theta$, such that $\rho(f(x),f(y)) \leq \alpha(\rho)\rho(x,y)$, for each $\rho \in \theta$ and all $x,y \in X$.

*Theorem* 18. *Let* $(X,\mathcal{U})$ *be a uniform space, which is* $\theta$-*complete for some subgage* $\theta$ *for* $\mathcal{U}$, *and let* $f\colon X \to X$ *be a function. If* $f$ *is a* $\theta$-*contraction then* $f$ *has a unique fixed point.*

*Proof.* By the standard proof of Banach's Contraction Principle for metric spaces, we get that, for each $\rho \in \theta$, $\rho(f^n(x),f^{n+1}(x)) \leq \alpha(\rho)^n\rho(x,f(x))$ and $\rho(f^n(x),f^m(x)) \leq \frac{\alpha(\rho)^n}{1-\alpha(\rho)} d(x,f(x))$, where $f^n(x)$ is the $n^{th}$ iterate of $f$. Since $\lim_n \alpha(\rho)^n = 0$ we then get that $\{f^n(x)\}$ is a $\rho$-Cauchy sequence, for each $\rho \in \theta$. Consequently, $\lim_n f^n(x) = x_\rho$ in $(X,\rho)$, for each $\rho \in \theta$.

Next, note that $f(x_\rho) = x_\rho$, for each $\rho \in \theta$ ($x_\rho = \lim_n f^n(x)$ implies $f(x_\rho) = \lim_n f^{n+1}(x) = x_\rho$ in $(X,\rho)$ because $f\colon (X,\rho) \to (X,\rho)$ is continuous, for each $\rho \in \theta$). Finally, note that $x_\rho = x_\mu$ for all $\rho,\mu \in \theta$ (i.e. $f$ has a unique fixed point): Suppose not; say $x_\rho \neq x_\mu$, for some $\rho,\mu \in \theta$. Pick $\rho' \in \theta$ such that $\rho'(x_\rho,x_\mu) > 0$. Then $\rho'(x_\rho,x_\mu) = \rho'(f(x_\rho), f(x_\mu)) \leq \alpha(\rho')\rho'(x_\rho,x_\mu)$, a contradiction.

The preceding result generalizes Theorem 2.3 of [5]. Consequently, several results from [5] can be generalized from sequentially complete spaces to sub-complete spaces.

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