THE MEASURE ON $S$-CLOSED SPACES

by

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1. Introduction

In functional analysis, $C(X)$ is a beautiful space, where $X = [0,1]$. Riesz gave a well-known conclusion--The dual of $C(X)$ is the space of all finite signed measures on $X$ with the norm defined by $||v|| = |v|(X)$. If $X$ is a compact and Hausdorff space, the same result can be obtained. Thompson [1] first introduced the concept of S-closed spaces. References [2-4] studied a series of topological properties of S-closed spaces. In this paper, a measure on S-closed spaces with certain properties is constructed. Some S-closed spaces are neither compact nor Hausdorff, but some interesting results can still be obtained. For example, if $X$ is a S-closed space, then to each bounded linear functional $F$ on $C(X)$, the set of all continuous real-valued functions on $X$, there corresponds exactly one finite signed $F$-S measure $v$ on $X$ such that $F(f) = \int fdv$, for each $f \in C(X)$ and $||F|| = |v|(X)$.

Let $X$ be a topological space; a set $P \subset X$ is called a regular closed set of $X$, if $P = P^O\cap P^\circ$, where $\circ$ and $-$ are the interior and the closure operations on $X$; a set $Q \subset X$ is called a regular open set of $X$, if $Q = Q^O\cap Q^\circ$. A topological space $X$ is said to be S-closed if every cover for $X$, consisting of regular closed sets, has a finite subcover.
Example 1. Let \( S = \{x: 0 < x < 1\} \) be the open unit interval. \( \tau = \{\phi\} \cup \{X\setminus A: A \subset X \text{ and } |A| \leq \omega_0\} \). Then \( X = (S, \tau) \) is a topological space.

Let \( A \subset X \). If \( A \) is countable, then \( A^O = \phi \); and if \( A \) is uncountable, then \( A^- = X \). Whence there are only two regular closed sets in \( X \). It is not hard to see that \( X \) is an S-closed \( T_1 \) space, but not a Hausdorff space; therefore, not a compact space either.

Let \( X \) be a topological space; \( A \subset X \) is said to be an S-closed set of \( X \) if every cover of regular closed sets in \( X \) for \( A \) has a finite subcover.

Proposition 1. The finite union of S-closed sets of a topological space is S-closed.

The proof is straightforward and is omitted.

Proposition 2. If \( P \) is a regular closed set of an S-closed space \( X \), then \( P \) is S-closed.

Proof. Let \( \{U_t: t \in T\} \) be a family of regular closed sets of \( X \), which covers \( P \). That is
\[
\bigcap\{X - U_t: t \in T\} \subset X - P \subset (X - P)^-.
\]
It follows from Theorem 4 in [3] that there exists a finite subfamily \( \{X - U_{t_1}, \ldots, X - U_{t_n}\} \) such that
\[
\bigcap_{t=1}^n (X - U_{t_i}) \subset (X - P)^-.
\]
From that \( P \) is a regular closed set it follows that
\[
(X - P)^O = X - P \quad \text{and that}
\[
\left[\bigcap_{i=1}^n (X - U_{t_i})\right]^O = \bigcap_{i=1}^n (X - U_{t_i}) \subset (X - P)^- = X - P.
\]
This implies that $P \subseteq \bigcup_{i=1}^{n} U_i$.

**Proposition 3.** If $g: X \to Y$ is a continuous mapping from an $S$-closed space $X$ into a metric space $Y$, then $g(X)$ is a bounded set of $Y$.

**Proof.** For every $x \in X$, choose a unit open ball $V_x = B(g(x), 1)$ of $g(x)$. The continuity of $g$ implies that $[g^{-1}(V_x)]^-$ is regular closed in $X$, and

$$\bigcup \{[g^{-1}(V_x)]^- : x \in X\} \supset\! X.$$ 

Since $X$ is an $S$-closed space, there exists a finite family $\{[g^{-1}(V_{x_i})]^- : i = 1, 2, \ldots, n\}$ such that

$$\bigcup_{i=1}^{n} [g^{-1}(V_{x_i})]^- \supset\! X.$$ 

It follows from the continuity of $g$ that

$$\bigcup_{i=1}^{n} V_{x_i}^- = [\bigcup_{i=1}^{n} g \circ g^{-1}(V_{x_i})]^- \supset\! g(X).$$ 

This implies that $g(X)$ is bounded.

A topological space $X$ is called a locally $S$-closed space if for every $x \in X$, there exists a neighborhood $U_x$ of the point $x$ such that $U_x^-$ is contained in an $S$-closed set of $X$.

**Proposition 4.** Every $S$-closed set of a $T_1$ space $X$ is closed.

**Proof.** Let $A$ be an $S$-closed set of $X$ and let $p$ be a point of $X \setminus A$. For every $x \in X \setminus \{p\}$, there exists a regular open neighborhood $U_x$ of the point $p$ such that $x \notin U_x$ and that $\cap\{U_x : x \in X \setminus \{p\}\} = \{p\}$. Hence

$$X \setminus \{p\} = \bigcup \{X \setminus U_x : x \in X \setminus \{p\}\} \supset A.$$
As $A$ is an $S$-closed set of $X$, there exists a finite family
\[ \{X \setminus U_{x_1}, X \setminus U_{x_2}, \ldots, X \setminus U_{x_k}\} \]
such that
\[ \bigcup_{j=1}^{k} (X \setminus U_{x_j}) \supseteq A. \]
Take $U(p) = \cap_{j=1}^{k} U_{x_j}$. Hence $U(p) \cap A = \emptyset$. That is $U(p) \subseteq X \setminus A$.

**Corollary.** Every $S$-closed set of a Hausdorff space is closed.

**Proposition 5.** Let $A$ be an $S$-closed set of a topological space $X$. If $G \subseteq A$ and $G$ is regular open in $X$, then $G$ is $S$-closed in $X$.

**Proof.** Let $\{U_s^- : s \in S\}$ be a family of regular closed sets of $X$ which covers $G$. Then $\{U_s^- : s \in S\} \cup \{X \setminus G\}$ is a cover of $A$ of regular closed sets. Since $A$ is $S$-closed in $X$, there exists a finite subcover $\{U_{s_1}^-, U_{s_2}^-, \ldots, U_{s_n}^-\} \cup \{X \setminus G\}$ for the set $A$. Hence $\{U_{s_1}^-, U_{s_2}^-, \ldots, U_{s_n}^-\}$ is a finite subcover for $G$.

2. The Measure on $S$-Closed Spaces

**Lemma 1.** Let $X$ be a locally $S$-closed $T_1$ space. Then for any $S$-closed set $A \subsetneq X$ there exists a both closed and open set $U \subsetneq X$ such that $A \subseteq U$ and $U$ is contained in an $S$-closed set of $X$.

**Proof.** For every $x \in A$, choose an open neighborhood $V_x$ of $x$ and an $S$-closed set $W_x$ of $X$ such that $V_x^- \subset W_x$. Pick a point $p \in X \setminus A$ and a regular open neighborhood $U_x$ of $x$ such that $p \notin U_x$. Let $Y_x = (V_x \cap U_x)^O$. Then $Y_x$ is regular open in $X$ with $p \notin Y_x \subset Y_x^- \subset W_x$. So by Proposition 5, $Y_x$ is $S$-closed in $X$. By Proposition 4, $Y_x$ is closed in
Thus \( \{X_x : x \in A\} \) is a family of regular closed sets which covers \( A \). From that \( A \) is S-closed in \( X \) it follows that there exists a finite subcover \( \{Y_{x_i} : i = 1,2,\cdots,n\} \) for \( A \). Then \( U = \bigcup_{i=1}^{n} Y_{x_i} \supseteq A \) is closed, open and S-closed in \( X \).

Let \( X \) be a topological space. Take \( C(X) \) to indicate the family of all real-valued continuous functions on \( X \). And define 

\[
C_0(X) = \{f \in C(X) : \text{there exists an S-closed set } A \text{ of } X \text{ such that } f(x) \neq 0 \text{ implies } x \in A\}.
\]

The class of \( F - S \) sets is defined to be the smallest \( \sigma \)-algebra \( B \) of subsets of \( X \) such that functions in \( C_0(X) \) are measurable with respect to \( B \). A measure \( \mu \) is called an \( F - S \) measure on \( X \), if its domain of definition is the \( \sigma \)-algebra \( B \) of \( F - S \) sets, and \( \mu(A) < \infty \) for each S-closed set \( A \) in \( B \).

**Lemma 2.** If \( X \) is a topological space, then \( C_0(X) \) is a vector lattice.

**Proof.** It suffices to show that \( \alpha f + \beta g \), \( f \lor g \) and \( f \land g \) belong to \( C_0(X) \), whenever \( f,g \in C_0(X) \) and \( \alpha, \beta \in \mathbb{R} \), the set of all real numbers. Since \( \{x \in X : (\alpha f + \beta g)(x) \neq 0\} \subseteq \{x \in X : f(x) \neq 0\} \cup \{x \in X : g(x) \neq 0\} \). It follows from Proposition 1 that \( \alpha f + \beta g \in C_0(X) \). For \( f \land g = f + g - (f \lor g) \) and \( f \lor g = (f - g) \lor 0 + g \), we only need to prove that if \( f \in C_0(X) \) then \( f \lor 0 \in C_0(X) \). Indeed, \( f \lor 0 \) is continuous and \( \{x : f(x) \neq 0\} \Rightarrow \{x : (f \lor 0)(x) \neq 0\} \). Hence, \( f \in C_0(X) \) implies \( f \lor 0 \in C_0(X) \).
Theorem 1. Let $X$ be a locally $S$-closed $T_\text{I}$ space, $I$ a positive linear functional on the set $C_0(X)$. Then there is an $F-S$ measure $\mu$ such that for each $f \in C_0(X)$, $I(f) = \int f \, d\mu$.

Proof. The set $C_0(X)$ is a vector lattice by Lemma 2. Now we show that $I$ is a Daniell integral on $C_0(X)$ (see [5]). To see this end, let $\zeta \in C_0(X)$ and $(\zeta_n)$ be an increasing sequence of functions in $C_0(X)$ such that
\[ \zeta \leq \lim \zeta_n. \]
We may assume that $\zeta$ and each $\zeta_n$ are non-negative. Take $K = \{x \in X: \zeta(x) \neq 0\}$. Then $K^-$ is $S$-closed in $X$. In fact, since $\zeta \in C_0(X)$, there exists an $S$-closed set $G$ of $X$ such that $G \supseteq K^-$. Proposition 5 implies that the regular open set $K^O$ is $S$-closed in $X$. So, Proposition 4 implies that $K^O$ is closed. That is $K^O = K^-$. So $K^-$ is $S$-closed in $X$.

Take a non-negative $g \in C_0(X)$ such that $g(x) = 1$, for each $x \in K^-$. By Lemma 1, this can be done.

For any given $\varepsilon > 0$, the set $K^-$ is covered by regular sets $\{O_n^-: n = 1, 2, \cdots\}$, where $O_n^- = \{x \in X: \zeta(x) - \varepsilon g(x) < \zeta_n(x)\}$. Since $K^-$ is $S$-closed in $X$, and $O_n^-$'s are increasing, there must be an $N$ such that $K^- \subseteq O_N^-$. Hence $\zeta - \varepsilon g < \zeta_N$ on $K$. Since $\zeta \equiv 0$ outside $K$, $\zeta - \varepsilon g \leq \zeta_N$ holds everywhere. So
\[ I(\zeta) - \varepsilon I(g) \leq I(\zeta_N) \leq \lim I(\zeta_n). \]
Since $\varepsilon$ was arbitrary and $I(g) < \infty$, it must be that
\[ I(\zeta) \leq \lim I(\zeta_n). \]
Thus $I$ is a Daniell integral.
It follows from [5] Stone Theorem that there is a measure \( \mu \) defined on the class \( B \) of \( F - S \) sets such that for each \( f \) in \( C_0(X) \),
\[
I(f) = \int fd\mu.
\]
It remains only to show that if \( K \) is an \( S \)-closed set in \( B \), then \( \mu(K) < \infty \). In fact, from Lemma 1 there exists \( h \in C_0(X) \) such that \( h(x) = 1 \), for each \( x \in K \), then \( \mu(K) \leq \int hdu = I(h) < \infty \).

**Theorem 2.** If \( X \) is an \( S \)-closed space and \( I \) a positive linear functional on \( C(X) \), then there is a unique \( F - S \) measure \( \mu \) on \( X \) such that \( I(f) = \int fd\mu \), for each \( f \in C(X) \).

**Proof.** It follows from the proof of Theorem 1 that it suffices to show that \( \mu \) is unique. Because \( 1 \in C(X) \), Theorem 20 in [5] implies the uniqueness.

**Theorem 3.** Let \( X \) be an \( S \)-closed space. Then to each bounded linear functional \( F \) on \( C(X) \), there corresponds a unique finite signed \( F - S \) measure \( \nu \) on \( X \) such that
\[
F(f) = \int fd\nu,
\]
for each \( f \in C(X) \). Moreover, \( ||F|| = ||\nu||(X) \).

**Proof.** By Proposition 3, \( C(X) \) is a normed linear space with the norm \( || \cdot || \) defined by \( ||f|| = \sup|f(x)| \), for each \( f \in C(X) \).

Let \( F = F_+ - F_- \) be defined as in [5] Proposition 23. Then by Theorem 2, there are finite \( F - S \) measures \( \mu_1 \) and \( \mu_2 \) such that
\[
F_+(f) = \int fd\mu_1 \quad \text{and} \quad F_-(f) = \int fd\mu_2,
\]
for each \( f \in C(X) \).
Set \( \nu = \mu_1 - \mu_2 \); then \( \nu \) is a finite signed \( F-S \) measure, and \( F(f) = \int f \, d\nu \), for each \( f \in C(X) \). Now, for each \( f \in C(X) \), \( |F(f)| \leq \int |f| \, d|\nu| \leq ||f|| \cdot |\nu|(X) \). Hence, \( ||F|| \leq |\nu|(X) \). But
\[
|\nu|(X) \leq \mu_1(X) + \mu_2(X)
= F^+(1) + F^-(1) = ||F||.
\]
Thus, \( ||F|| = |\nu|(X) \).

To show the uniqueness of \( \nu \), let \( \nu_1 \) and \( \nu_2 \) be two finite signed \( F-S \) measures on \( X \) such that \( \int f \, d\nu_i = F(f), \ i = 1, 2. \)

Then \( \lambda = \nu_1 - \nu_2 \) would be a finite signed \( F-S \) measure on \( X \) such that \( \int f \, d\lambda = 0 \), for each \( f \in C(X) \). Let \( \lambda = \lambda^+ - \lambda^- \) be the Jordan decomposition of \( \lambda \). Then the integration with respect to \( \lambda^+ \) gives the same positive linear functional on \( C(X) \) as that given by \( \lambda^- \); and by Theorem 2, it must be \( \lambda^+ = \lambda^- \). Hence \( \lambda = 0 \) and \( \nu_1 = \nu_2 \).

**Theorem 4.** Let \( X \) be an \( S \)-closed space. Then to each bounded functional \( F \) on \( C(X) \) and \( 0 < p < +\infty \), there corresponds one finite \( F-S \) measure \( U \) on \( X \) such that for each \( f \in C(X) \), \( F(f) = (\int |f|^p \, dU)^{1/p} \) if and only if there exists a unique positive linear functional \( I \) on \( C(X) \) such that \( F^P(f) = I(|f|^p) \), for each \( f \in C(X) \). Moreover, \( U(X) = F^P(1) \).

The proof is straightforward and is omitted.

We conclude this paper with a problem: Let \( X \) be an \( S \)-closed \( T_1 \) space; then the dual of \( C(X) \) is (isometrically isomorphic to) the space of all finite signed \( F-S \) measures on \( X \) with the norm defined by \( ||\nu|| = |\nu|(X) \).
References


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