Research Announcement:
NON-HOMOGENEITY OF
INTERMEDIATE UNIVERSAL CONTINUA

by
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For each pair of non-negative integers \( n \) and \( k, \ k > n, \) Menger [6] has described an \( n \)-dimensional continuum \( M^k_n \) in \( E^k \) which is universal with respect to containing homeomorphic copies of every \( n \)-dimensional continuum which can be embedded in \( E^k \). For any \( k > 0, M^k_0 \) is the standard Cantor set. \( M^2_1 \) is the Sierpiński universal plane curve [8], [9], and \( M^k_1(k > 3) \) is the Menger universal curve.

The Cantor set is clearly homogeneous. It is a classical result that the Sierpiński curve is not homogeneous. Anderson [1] proved in 1958 that the Menger universal curve is homogeneous and gave a characterization of the continuum.

In early 1983 the author announced informally [5] that \( M^k_n \) is never homogeneous for \( n > 0, \ k < 2n + 1, \) and indicated an argument for the result. The result was not written up for publication at the time since the question of homogeneity for the case \( k \geq 2n + 1, \) when no embedding restrictions apply to the universality of \( M^k_n, \) remained open. Since then, Bestvina [2] has given a very nice proof of the homogeneity of \( M^k_n \) for \( k \geq 2n + 1 \) and a complete characterization of such spaces. Several persons have expressed interest in the result announced earlier for \( k < 2n + 1, \) or uncertainty about the status of this problem. It is the purpose of this note to clarify this.

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Alternative constructions of universal continua have been given by Lefschetz [4] and Pasynkov [7]. The space $M^k_n$ in Lefschetz' construction is the intersection of a nested sequence of $k$-manifolds with boundary. Let $M^k_{(0,n)}$ be a $k$-simplex. Inductively, let $M^k_{(i+1,n)}$ be the star of the $n$-skeleton of $M^k_{(i,n)}$ in the second barycentric subdivision of $M^k_{(i,n)}$. Set $M^k_n = \bigcap_{i=0}^{\infty} M^k_{(i,n)}$. It remains an open problem to characterize $M^k_n$ for $k < 2n + 1$ as well as to determine whether the constructions of Lefschetz, Menger, and Pasynkov yield homeomorphic continua. (Some results are known for the case $k = n + 1$ [3],[9].) Our argument is described in terms of Lefschetz' construction, but can be appropriately modified for the other two.

**Theorem.** $M^k_n$ is not homogeneous for $n > 0$, $k < 2n + 1$.

**Sketch of proof.** The case $k = n + 1$ is already known from arguments similar to that for the Sierpiński curve, i.e. local separation vs. its lack. However, this is also covered by the general case, which uses a generalization of these arguments.

Let $B = \{x \in M^k_n | x$ is in the boundary of some $M^k_{(i,n)}$ in the defining sequence for $M^k_n\}$, and

let $I = \{x \in M^k_n | x$ is in the interior of every $M^k_{(i,n)}$ in the defining sequence for $M^k_n\}.$

There are two alternative but closely related methods to topologically distinguish between points in $B$ and points in $I$.

The first involves local linking properties. If $x \in B$, there is an $n$-sphere $\Sigma$ in $M^k_n$ containing $x$ (and a subset of
the boundary of some $M_{(i,n)}^k$ such that every $(k - n - 1)$-sphere in $M_{n}^k - \Sigma$ is null-homotopic in $M_{n}^k - \Sigma$. No point in I has this property. If $y \in I$ and $S$ is any $n$-sphere in $M_{n}^k$ containing $y$, there is a $(k - n - 1)$-sphere $T$ in $M_{n}^k - S$ such that every null homotopy of $T$ in $M_{n}^k$ intersects $S$.

(One can if desired restrict one's attention to $S$ being as nice as desired in terms of ulc properties, etc., since it is to be compared to the $\Sigma$ containing $x$, which can be chosen to have any such desired properties.)

Alternatively, one can use a variation on the disjoint disk property. If $x \in B$, there is an embedding $f: C^n \to M_n^k$ with $f(0) = x$, where $C^n$ is an $n$-simplex and 0 its barycenter, such that for any map $g: C^{k-n} \to M_n^k$ and any $\varepsilon > 0$ there are maps $\tilde{f}: C^n \to M_n^k$ and $\tilde{g}: C^{k-n} \to M_n^k$ with $\text{dist}(f, \tilde{f}) < \varepsilon$ and $\text{dist}(g, \tilde{g}) < \varepsilon$, such that $\tilde{f}(C^n) \cap \tilde{g}(C^{k-n}) = \emptyset$.

No point of I has this property. If $y \in I$ and $h: C^n \to M_n^k$ is any embedding with $h(0) = y$, when there exist $j: C^{k-n} \to M_n^k$ and $\varepsilon > 0$, such that if $\tilde{h}: C^n \to M_n^k$ and $\tilde{j}: C^{k-n} \to M_n^k$ are two maps with $\text{dist}(h, \tilde{h}) < \varepsilon$ and $\text{dist}(j, \tilde{j}) < \varepsilon$ then $h(C^n) \cap j(C^{k-n}) \neq \emptyset$.

For the first argument, the non-existence of null-homotopies of $n$ spheres in any $M_n^k$ prevents it from applying to $M_n^{2n+1}$. For the second argument, the fact that $M_n^{2n+1}$ has the disjoint $n$-cell property, and $k - n > n$ in this case, prevents the argument from applying for $k > 2n$. It is the fact that $M_n^{2n+1}$ satisfies the disjoint $n$-cell property which is central in the characterization of $M_n^{2n+1}$ and the proof of its homogeneity.
Since the above arguments distinguish points based on local properties, they also show that no Sierpiński manifold (i.e. continuum locally homeomorphic to $M_n^k$ for some $n > 0$, $n < k < 2n + 1$) is homogeneous. Left open are the questions of how many orbits $M_n^k$ has under the action of its homeomorphism group and whether either of the sets $B$ or $I$ constitutes an orbit, as well as homogeneity properties of $M_n^k$ under classes of maps more general than homeomorphisms.

References


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