COMPOSANTS OF INDECOMPOSABLE
STONE-CECH REMAINDERS

by

DAVID P. BELLAMY
COMPOSANTS OF INDECOMPOSABLE STONE-ČECH REMAINDERS

David P. Bellamy

This article concerns the properties of certain spaces which can occur as Stone-Čech remainders of locally compact Hausdorff spaces. I want to thank B. Diamond for two very useful conversations on this topic, and for calling to my attention Lemma 1 which made this work possible.

A continuum is a compact, connected Hausdorff space. Let \( Y \) be a continuum. \( Y \) is irreducible between the points \( a, b \in Y \) if no proper subcontinuum of \( Y \) contains both of them. This will be denoted by \( Y = [a, b] \), with the understanding that if \( a, b \in \mathbb{R} \), the usual meaning applies. \( Y \) is connected im Kleinen at \( p \in Y \) provided every neighborhood of \( p \) contains a neighborhood of \( p \) which is connected and closed in \( Y \). \( Y \) is indecomposable if it is not the union of two of its proper subcontinua, or equivalently if every proper subcontinuum of \( Y \) is nowhere dense. If \( p \in Y \), the composant of \( p \) in \( Y \), denoted \( C(Y;p) \), is defined by
\[
C(Y;p) = \{ y \in Y | Y \neq [p,y] \}.
\]

\( C(Y;p) \) is then the union of all the proper subcontinua of \( Y \) containing \( p \). If \( Y \) is nondegenerate and indecomposable, the sets \( C(Y;p) \) partition \( Y \); that is, \( y \in C(Y;p) \) is an equivalence relation. \( C(Y) \) will denote the set of composants of \( Y \). Nondegenerate metrizable indecomposable continua have been known since the 1920's to have exactly \( c \) composants.
For the nonmetric case, the situation is more complicated. It is known that there exist indecomposable continua \( X \) such that \( \mathcal{C}(X) \) has cardinality 1, 2 or \( 2^m \) for any infinite cardinal number \( m \) [3], [12]. Whether other numbers are possible is open.

For any completely regular space \( X \), \( \beta X \) will denote its Stone-Čech compactification and \( X^* \) will denote the remainder \( \beta X - X \). A will always denote \((0,1]\) and \( I \) will denote \([0,1]\). \( A^* \) is an indecomposable continuum [1], [2] or [13], but the cardinality of \( \mathcal{C}(A^*) \) depends on your set theory; it is known that it can be either one or \( 2^c \) [5], [10], [11]. The purpose of this paper is to show that for many other non-pseudocompact \( X \) with \( X^* \) an indecomposable continuum \( \mathcal{C}(X^*) \) and \( \mathcal{C}(A^*) \) are equipollent.

Dickman [7] showed that a half open interval is essentially the only locally connected and locally compact metric space with an indecomposable continuum as its Stone-Čech remainder; however, L. R. Rubin and the author demonstrated the existence of a broader class of objects, called waves, with this property [4].

**Definition.** A wave from \( a \) to \( b \) is a topological pair \((Y,X)\) such that \( Y \) is a continuum irreducible between \( a \) and \( b \), \( Y \) is both connected in Kleinen and first countable at \( b \), and \( X = Y - \{b\} \).

**Theorem 1** [4]. If \((Y,X)\) is a wave from \( a \) to \( b \), then \( X^* \) is an indecomposable continuum.
An indecomposable continuum of this type will be called a wave remainder.

Theorem 2. There exist wave remainders of arbitrarily large cardinality.

Proof. Given a limit ordinal number \( m \), perform a long line construction on the ordinal \( \alpha = m \times \omega \). That is, define \( X \) to be the set \( \alpha \times [0,1) \) with the lexicographic order topology, and let \( Y \) be the one point compactification of \( X \). Let \( S_0 \) denote the closure of the subset of \( X \), \( \{(\beta,t) | \beta < m\} \), and let \( S \) denote \( S_0 \) with its top and bottom points identified. It is easy to see, using a spiral-like construction in \( Y \times S \), that \( X \) has a compactification with remainder \( S \), and since \( \beta X - X \) admits a continuous map onto \( S \), it has cardinality at least as large as \( S \). \( S \), however, has cardinality at least that of \( m \), so the proof is done.

The principal result here is:

Theorem 3. If \( X^* \) is any wave remainder, then \( C(A^*) \) and \( C(x^*) \) are equipollent.

To prove this, a number of Lemmas are needed.

Lemma 1. Let \( X \) and \( Y \) be completely regular spaces and let \( f: X \to Y \) be a monotone quotient map. Then \( \beta f: \beta X \to \beta Y \) is a monotone map also.

Proof. This is a special case of B. Diamond's theorem 4.7 of [6, p. 76].
Lemma 2. If $X$ is a locally compact space and $D$ is a decomposition of $X$ into compact sets such that the nondegenerate members of $D$ form a neighborhood finite collection, then the quotient map $q: X \to X/D$ is perfect. Consequently, $\beta q(X^*) = (X/D)^*$ and $(\beta q)^*(X/D)^* = X^*$.

Proof. Each point inverse is clearly compact, and $D$ is upper semicontinuous, since if $A \in D$ and $U$ is open with $A \subseteq U$, $U - U\{B \in D|B \neq A \text{ and } B \text{ is nondegenerate}\}$ is a saturated open set containing $A$ and contained in $U$. The last sentence follows from Lemma 1.5 of [8, p. 87] and the definition of compactification. (Henriksen and Isbell use the term fitting map for what is nowadays commonly called a perfect map.)

Lemma 3. Suppose $S$ and $Z$ are indecomposable continua, $Z$ is nondegenerate, and $f: S \to Z$ is a monotone onto map. Then $f$ induces a bijection between $C(S)$ and $C(Z)$.

Proof. Let $C$ be any composant of $Z$. Then

$$C = \bigcup \{W|p \in W, W \text{ a proper subcontinuum of } Z\}$$

for some $p \in Z$. Thus,

$$f^+(C) = \bigcup \{f^+(W)|p \in W, W \text{ a proper subcontinuum of } Z\}.$$

Since for $W \neq Z$, $f^+(W) \neq S$, it follows that $f^+(C)$ is a subset of a single composant of $S$. Define $H: C(Z) \to C(S)$ by $H(C) = \text{the composant of } S \text{ containing } f^+(C)$. Then, if $x \in S$, $f(x) \in Z$ and thus $f(x) \in D$ for some composant $D$ of $Z$. Thus, $f^+(D) \subseteq C(S;x)$, so that $H$ is surjective.

Suppose $W$ is a proper subcontinuum of $S$ and that $f(W) = Z$. Since $Z$ is nondegenerate, there is a nonempty,
nondense open $U \subseteq Z$. $W$ is nowhere dense in $S$, so neither $f^+(U)$ nor $f^+(Z-U)$ is a subset of $W$. By monotonicity, $W \cup f^+(Z-U)$ and $W \cup f^+(\overline{U})$ are proper subcontinua of $S$ whose union is $S$, a contradiction to the indecomposability of $S$. Consequently, for each proper subcontinuum $W$ of $S$, $f(W) \neq Z$.

Therefore, for any $D \in \mathcal{C}(S)$, $f(D)$ is contained in a single composant of $Z$. (Since $f$ commutes with unions, the same argument works as for $f^+$ above.) If for two composants $C_1$ and $C_2$ of $Z$, $H(C_1) = H(C_2)$, then $f^+(C_1 \cup C_2) \subseteq H(C_1)$ and so $C_1 \cup C_2 \subseteq f(H(C_1)) \subseteq C_3$ for some single composant $C_3$ of $Z$. This is possible only if $C_1 = C_2 = C_3$; therefore, $H$ is injective and hence bijective.

Definition. A wave $(Y,X)$ from $a$ to $b$ has a cofinal sequence of cutpoints provided that there is a sequence $(b_n)_{n=0}^\infty$ converging to $b$ such that $b_0 = a$ and for each $n \geq 1$, $b_n$ separates $b_{n-1}$ from $b$.

Remark. This is a fairly strong property. It is easy to string together a sequence of indecomposable continua with more than three composants to form a wave in which no connected, nowhere dense set separates.

Lemma 4. Let $(Y,X)$ be a wave from $a$ to $b$. Then there is a descending sequence of continua $(W_i)_{i=0}^\infty$ such that $W_0 = Y \neq W_i$; for each $i \geq 1$, $W_i \subseteq \text{Int}(W_{i-1})$; and $\cap_{i=0}^\infty W_i = \{b\}$. The $W_i$'s, $i \geq 1$, can be chosen to have one-point boundaries if and only if $(Y,X)$ has a cofinal sequence of cutpoints.
Proof. First countability and connectedness in Kleinen at b enable one to do a simple recursive construction of the \( W_i \)'s. Irreducibility is used for the last sentence.

Convention. If \( (Y,X) \) is a wave from a to b and \( D \) is an upper-semicontinuous decomposition of \( Y \) with \( \{b\} \) a degenerate element of \( D \), then \( \frac{X}{D} \) will be used to denote the image of \( X \) under the quotient map \( Y \to \frac{Y}{D} \). \( D - \{b\} \) is an upper-semicontinuous decomposition of \( X \) in this case.

Lemma 5. Let \( (Y,X) \) be a wave from a to b. Then there is a monotone decomposition \( D \) of \( Y \) such that every nondegenerate member of \( D \) is a subset of \( X \), the nondegenerate members of \( D \) form a neighborhood finite family in \( X \), and \( \left( \frac{Y}{D}, \frac{X}{D} \right) \) is a wave from \([a]\) to \([b]\) with a cofinal sequence of cutpoints.

Proof. Define \( D_n = \overline{W_n} - \overline{W_{n+1}} \), and let \( D \) be the decomposition with nondegenerate elements \( \{D_n\}_{n \text{ odd}} \). By irreducibility, each \( D_n \) is connected and becomes a cutpoint of the quotient \( \frac{Y}{D} \), as required. Both \( \{a\} \) and \( \{b\} \) are degenerate elements of \( D \), making the necessary verifications easy.

Lemma 6. Let \( (Y,X) \) be a wave from a to b and let \( D \) be an upper semicontinuous monotone decomposition of \( (Y,X) \) with the nondegenerate elements forming a neighborhood finite collection in \( X \). Then \( X^* \) has the same number of composants as \( \left( \frac{X}{D} \right)^* \).
Proof. If \( q: X \to \frac{X}{D} \) is the quotient map, then
\( \beta q(X^*) = \left( \frac{X}{D} \right)^* \) and \( (\beta q)^* \left( \left( \frac{X}{D} \right)^* \right) = X^* \), and thus \( (\beta q)|_{X^*} \) is monotone by Lemma 1. By Lemma 3, \( (\beta q)|_{X^*} \) induces a bijection between the set of composants of \( X^* \) and that of \( \left( \frac{X}{D} \right)^* \), completing the argument.

Definition. A wave \((Y,X)\) from \(a\) to \(b\) has a cofinal sequence of closed intervals if there is a descending sequence of continua, \( \{W_i\}_{i=0}^\infty \) with \( W_0 = Y \neq W_1 \); for each \( i \geq 1 \), \( W_i \subseteq \text{Int}(W_{i-1}) \); \( \cap_{i=0}^\infty W_i = \{b\} \); and for each odd \( i \), \( W_i - W_{i+1} \) is homeomorphic to \( I \).

Lemma 7. Let \((Y,X)\) be a wave from \(a\) to \(b\), with a cofinal sequence of cutpoints. Then there is a bijection between \( C(X^*) \) and \( C(A^*) \).

Proof. Suppose \( \{W_i\}_{i=0}^\infty \) is a descending sequence of continua in \( Y \) such that \( W_0 = Y \), and for \( i \geq 1 \), \( W_i \) has boundary \( \{b_i\} \), and \( \cap_{i=0}^\infty W_i = \{b\} \). Define \( L_1 \subseteq Y \) by \( L_1 = \overline{W_{i-1} - W_i} \). Now, define \( \hat{X} \subseteq Y \times I \) by
\[
\hat{X} = \left( \bigcup_{i=1}^{\infty} \left( L_i \times \left[ \frac{1}{i}, \frac{1}{i+1} \right] \right) \right) \cup \left( \bigcup_{i=1}^{\infty} \left( \{b_i\} \times \left[ \frac{1}{i+1}, \frac{1}{i} \right] \right) \right).
\]
The only limit point of \( \hat{X} \) which does not belong to \( \hat{X} \) is \((b,0)\). Thus, if \( \hat{Y} = \hat{X} \cup \{(b,0)\} \), it is easy to see that \((\hat{Y},\hat{X})\) is a wave from \((a,1)\) to \((b,0)\) with a cofinal sequence of closed intervals, \( \left( \{b_i\} \times \left[ \frac{1}{i+1}, \frac{1}{i} \right] \right)_{i=1}^\infty \). Shrinking each of them to a point is accomplished by restricting the projection \( Y \times I \to Y \) to \( \hat{Y} \), so that the quotient of \((\hat{Y},\hat{X})\) so obtained is \((Y,X)\).

Thus \( C(\hat{X}^*) \) and \( C(X^*) \) are equipollent.
Continuing, the projection $Y \times I \to I$ restricted to $\hat{Y}$ is also monotone and has the effect of shrinking each $L_i \times \{ \frac{1}{2} \}$ to a point. Thus, $(\hat{Y}, \hat{X})$ also admits a monotone quotient map onto $(I, A)$, so that $C(A^*)$ and $C(\hat{X}^*)$ are also equipollent. Thus, the set of composants of $X^*$ and that of $A^*$ are also equipollent, by transitivity.

Proof of Theorem 3. This is now immediate from Lemmas 5, 6, and 7.

References

5. A. Blass, Near coherence of filters, II: Applications to operator ideals, the Stone-Čech remainder of a half-line, order ideals of sequences, and slenderness of groups (Preprint).


University of Delaware

Newark, Delaware 19716