DISPERSION POINTS AND FIXED POINTS OF SUBSETS OF THE PLANE

by

Andrzej Gutek
During the Spring Topology Conference in 1986 Hidefumi Katsuura asked whether there is a connected subset $X$ of the plane with the dispersion point $p$ such that for some non-constant function $f$ from $X$ into itself the point $p$ is not the fixed point of $f$. He also asked whether the function $f$ can be onto. We answer both of these questions in affirmative.

**Definition.** A point $p$ in a connected topological space $X$ is said to be a dispersion point of $X$ if each component of $X \setminus \{p\}$ consists of a single element, i.e. if $X \setminus \{p\}$ is totally disconnected.

**Definition.** If $f$ is a continuous function from a space $X$ into itself then a point $x$ of $X$ is said to be a fixed point of $f$ if $f(x) = x$.

Connected spaces with dispersion points were first defined by Knaster and Kuratowski in [K.K], and were extensively studied by Duda in [D]. In [C.V.] Cobb and Voxman asked whether the dispersion point was a fixed point of any non-constant function $f$ defined on a connected space with a dispersion point. In [K] Katsuura described a space $X$ with a dispersion point $p$ and a continuous non-constant mapping $f$ on $X$ such that $p$ is not a fixed point.
We modify Katsuura's construction to obtain such an
example in the plane. We show that function f may be onto.
In the construction we use the following theorem by
Katsuura:

**Theorem [K].** Suppose X is a totally disconnected
space, and \{Y(i): i ∈ I\} the collection of all quasi-
components of X. Let F be a proper closed subset of X that
has a point in common with every quasi-component. Let q
be the quotient map from X onto X/F. Then X/F is a con-
nected space with the dispersion point q(F).

**Example 1.** Let Q denote the set of rational numbers,
let R denote the set of real numbers. Let C be the Cantor
ternary set in the interval [0,1], i.e. C = \{∞
\sum_{n=1}^{∞} \frac{a_n}{3^n}: a_n = 0,2 \text{ and } n = 1,2,3,\cdots\}. If A is a subset of R and b
is a real number, then b + A = \{b + a: a ∈ A\} and
b * A = \{b * a: a ∈ A\}. If A is a subset of the plane and
(x,y) is a pair of numbers then (x,y) + A = \{(x + a,y + b):
(a,b) ∈ A\} and (x,y) * A = \{(xa,yb): (a,b) ∈ A\}.

Let d be a real number and let D = \{(c,d): c ∈ C\}.
For any point (u,d) in the plane and (c,d) in D let
s⁺((u,d);(c,d)) = \{(c + |c - u|\cos t, d + |c - u|\sin t):
0 ≤ t ≤ π \text{ and } t = c + q \text{ for some } q \text{ in } Q\} and
s⁻((u,d);(c,d)) = \{(c + |c - u|\cos t, d + |c - u|\sin t):
-π ≤ t ≤ 0 \text{ and } t = c + q \text{ for some } q \text{ in } Q\}.

We put S⁺((u,d);D) = \cup\{S⁺((u,d);(c,d)): (c,d) ∈ D\} and
S⁻((u,d);D) = \cup\{S⁻((u,d);(c,d)): (c,d) ∈ D\}.

For any real number d let [a,b](d) denote the set [a,b] ∩ Q
if d is a rational number, and [a,b]¬Q if d is an irrational
number. Put $S(0) = \bigcup \{ \{c\} \times [0,1](c): c \in C\} \cup C \times \{0\} \cup C \times \{1\}$. Let $C(1,i) = \frac{8+2i}{27} + \frac{1}{27} \ast C$ and let $S(1,i) = \bigcup \{ \{c\} \times \left[\frac{1}{2},3\right](c): c \in C(1,i)\}$, where $i = 1,2,3,4$.

Let $S(1) = S(1,1) \cup S(1,2) \cup S(1,3) \cup S(1,4) \cup S^\ast((\frac{31}{54},3);C(1,1) \times \{3\}) \cup S^\ast((\frac{23}{54},\frac{1}{2});C(1,1) \times \{\frac{1}{2}\}) \cup S^\ast((\frac{3}{54},\frac{1}{2});C(1,4) \times \{\frac{1}{2}\}) \cup S^\ast((\frac{39}{54},3);C(1,4) \times \{3\})$, (see figure 1).

For convenience we write $u(1,i) = \frac{7+6i}{54}$, $i = 1,2,3,4$.

In order to obtain $S(2)$ we repeat the construction of $S(1)$ for the sets $C \cap [0,3^{-n}]$ and $C \cap [\frac{2}{3},1]$ and replace in that construction the segments $[2^{-1},3]$ by the segments $[2^{-2},3]$. The figure 2 shows the set $S(0) \cup S(1) \cup S(2)$.

Formal description of $S(n)$, $n > 1$, is as follows. Put

$C(n,i) = 3^{-n} \ast C(n-1,i)$ if $i = 1,2,\ldots,2^n$, and

$c(n,i) = \frac{2}{3} + 3^{-n} \ast C(n-1,i-2^n)$ if $i = 2^n+1,\ldots,2^{n+1}$.

Let $S(n,i) = \bigcup \{ \{c\} \times [2^{-n},3](c): c \in C(n,i)\}$, where $i = 1,2,\ldots,2^{n+1}$.

Let $u(n,i) = 3^{-n} \ast u(n-1,i)$ if $i = 1,2,\ldots,2^n$, and

$u(n,i) = \frac{2}{3} + 3^{-n} u(n-1,i)$ if $i = 2^n+1,\ldots,2^{n+1}$.

Let $S(n) = \bigcup \{ S(n,i): i = 1,2,\ldots,2^{n+1}\} \cup S^\ast((u(n,1),3);C(n,1) \times \{3\}) \cup S^\ast((u(n,2),2^{-n});C(n,1) \times \{2^{-n}\}) \cup S^\ast((u(n,3),2^{-n});C(n,4) \times \{3\}) \cup \cdots \cup S^\ast((u(n,2^{n+1}),3);C(n,2^{n+1}) \times \{3\})$.

Let $X = \bigcup \{ S(n): n = 0,1,2,\ldots\}$. Observe that any quasi-component $K(c)$ of $X$ is the union of a segment-like set $\{c\} \times I(c)$ and $\sin(\frac{1}{x})$-like curve emerging from $(\frac{4}{9} + \frac{1}{9} c,3)$, where $c \in C$. By the theorem of Katsuura the quotient $Y = X/C \times \{0\}$ is a connected space and $q(C \times \{0\})$
FIGURE 1
is the dispersion point. By \(q\) we denote the quotient map from \(X\) onto \(Y\).

Let \(g\) be a linear and order-preserving mapping from \(C(n,i)\) onto \([0,3^{-n}] \cap C\) if \(i \equiv 2 (\text{mod} 4)\), and onto \([\frac{2}{3^n}, \frac{3}{3^n}] \cap C\) if \(i \equiv 3 (\text{mod} 4)\), and let \(g\) be a linear and order-reversing mapping from \(C(n,i)\) onto \([0,3^{-n}] \cap C\) if \(i \equiv 1 (\text{mod} 4)\), and onto \([\frac{2}{3^n}, \frac{3}{3^n}] \cap C\) if \(i \equiv 0 (\text{mod} 4)\). Let the map \(f\) from \(X\) into itself be defined as follows:

\[
\begin{align*}
  f(x) &= (0,1) \text{ if } x \in S(0), \\
  f(a,b) &= (0,0) \text{ if } b \geq \frac{5}{2}, \\
  f(a,b) &= (g(a), \frac{5}{2} - b) \text{ if } \frac{3}{2} < b < \frac{5}{2}, \\
  f(a,b) &= (g(a),1) \text{ if } (a,b) \in S(n,i) \text{ and } b \leq \frac{3}{2}, \\
  f(x) &= (g(c),1) \text{ if } x \in S^{-}((u(n,i),2^{-n});(c,2^{-n})) \text{ for some } c \in C(n,i).
\end{align*}
\]

Let \(g_q\) denote a map from \(Y\) into itself induced by \(f\). The map \(g_q\) is a continuous and non-constant function, and the dispersion point is not a fixed point of the map. The proof of continuity is straightforward but tedious.

**Example 2.** We modify the example 1 to obtain a mapping onto. Let \(f\) and \(X\) have the same meaning as in the example 1. For any point \(c\) in the Cantor set \(C\) let \(D(c)\) denote the set of all the points on the segment joining \(\left(\frac{4}{9} + \frac{1}{9} c, 4\right)\) and \((c,5)\) the second coordinate of which is rational if \(c\) is rational, and irrational if \(c\) is likewise. Let

\[
X(0) = X \cup \bigcup \{D(c) : c \in C\} \cup \bigcup \{\{c\} \times [3,4](c) : c \in C(1,2) \cup C(1,3)\} \text{ (see figure 3).}
\]

Let \(X(n) = (0,5) + X(n-1)\) for \(n = 1, 2, 3, \ldots\). Put \(X(\infty) = \bigcup \{X(n) : n = 0, 1, 2, \ldots\}\). Let \(F\) be a mapping from \(X(\infty)\)
into itself defined by
\[ F|X = f \]
\[ F(x) = (0,0) \text{ if } x \in X(0) \setminus X \]
\[ F(x) = x - (0,5) \text{ if } x \in X(n), \quad n = 1, 2, 3, \ldots \]

It is easy to see that \( F \) is onto.

Let \( Z \) be the quotient space \( X(\infty)/C \times \{0\} \), let \( q \) be the quotient map from \( X(\infty) \) onto \( Z \) and let \( F_q \) be the function on \( Z \) induced by \( F \). Observe that \( Z \) is a connected subset of the plane with the dispersion point \( q(C \times \{0\}) \), \( F_q \) is a continuous function from \( Z \) onto itself, and the dispersion point is not a fixed point of \( F_q \).

References


Tennessee Technological University

Cookeville, Tennessee 38505