DISPERSION POINTS AND FIXED POINTS OF SUBSETS OF THE PLANE

by

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During the Spring Topology Conference in 1986 Hidefumi Katsuura asked whether there is a connected subset \( X \) of the plane with the dispersion point \( p \) such that for some non-constant function \( f \) from \( X \) into itself the point \( p \) is not the fixed point of \( f \). He also asked whether the function \( f \) can be onto. We answer both of these questions in affirmative.

**Definition.** A point \( p \) in a connected topological space \( X \) is said to be a dispersion point of \( X \) if each component of \( X \setminus \{p\} \) consists of a single element, i.e. if \( X \setminus \{p\} \) is totally disconnected.

**Definition.** If \( f \) is a continuous function from a space \( X \) into itself then a point \( x \) of \( X \) is said to be a fixed point of \( f \) if \( f(x) = x \).

Connected spaces with dispersion points were first defined by Knaster and Kuratowski in [K.K], and were extensively studied by Duda in [D]. In [C.V.] Cobb and Voxman asked whether the dispersion point was a fixed point of any non-constant function \( f \) defined on a connected space with a dispersion point. In [K] Katsuura described a space \( X \) with a dispersion point \( p \) and a continuous non-constant mapping \( f \) on \( X \) such that \( p \) is not a fixed point.
We modify Katsuura's construction to obtain such an example in the plane. We show that function $f$ may be onto.

In the construction we use the following theorem by Katsuura:

Theorem [K]. Suppose $X$ is a totally disconnected space, and $\{Y(i) : i \in I\}$ the collection of all quasi-components of $X$. Let $F$ be a proper closed subset of $X$ that has a point in common with every quasi-component. Let $q$ be the quotient map from $X$ onto $X/F$. Then $X/F$ is a connected space with the dispersion point $q(F)$.

Example 1. Let $Q$ denote the set of rational numbers, let $R$ denote the set of real numbers. Let $C$ be the Cantor ternary set in the interval $[0,1]$, i.e. $C = \{ \sum_{n=1}^{\infty} \frac{a_n}{3^n} : a_n = 0,2$ and $n = 1,2,3,\ldots \}$. If $A$ is a subset of $R$ and $b$ is a real number, then $b + A = \{ b + a : a \in A \}$ and $b \cdot A = \{ b \cdot a : a \in A \}$. If $A$ is a subset of the plane and $(x,y)$ is a pair of numbers then $(x,y) + A = \{ (x + a, y + b) : (a,b) \in A \}$ and $(x,y) \cdot A = \{ (xa,yb) : (a,b) \in A \}$.

Let $d$ be a real number and let $D = \{ (c,d) : c \in C \}$.

For any point $(u,d)$ in the plane and $(c,d)$ in $D$ let

$S^+((u,d);(c,d)) = \{ (c + |c - u| \cos t, d + |c - u| \sin t) : 0 \leq t \leq \pi \text{ and } t = c + q \text{ for some } q \text{ in } Q \}$ and

$S^-((u,d);(c,d)) = \{ (c + |c - u| \cos t, d + |c - u| \sin t) : -\pi \leq t \leq 0 \text{ and } t = c + q \text{ for some } q \text{ in } Q \}$.

We put $S^+((u,d);D) = \cup \{ S^+(u,d);(c,d)) : (c,d) \in D \}$ and

$S^-((u,d);D) = \cup \{ S^-(u,d);(c,d)) : (c,d) \in D \}$.

For any real number $d$ let $[a,b](d)$ denote the set $[a,b] \cap Q$ if $d$ is a rational number, and $[a,b] \cap Q$ if $d$ is an irrational
number. Put $S(0) = \cup \{\{c\} \times [0,1]: c \in C\} \cup C \times \{0\} \cup C \times \{1\}$. Let $C(1,i) = \frac{8+2i}{27} + \frac{1}{27} \ast C$ and let $S(1,i) = \cup \{\{c\} \times [\frac{1}{2},3]: c \in C(1,i)\}$, where $i = 1,2,3,4$. Let $S(1) = S(1,1) \cup S(1,2) \cup S(1,3) \cup S(1,4) \cup S^+((\frac{15}{54},3); [C(1,1) \times \{3\}] \cup S^-((\frac{23}{54},1); C(1,1) \times \{\frac{1}{2}\}) \cup S^-((\frac{31}{54},1); C(1,4) \times \{\frac{1}{2}\}) \cup S^+((\frac{39}{54},3); C(1,4) \times \{3\})$.

For convenience we write $u(1,i) = \frac{7+6i}{54}$, $i = 1,2,3,4$.

In order to obtain $S(2)$ we repeat the construction of $S(1)$ for the sets $C \cap [0,\frac{3}{2}^{-n}]$ and $C \cap [\frac{2}{3},1]$ and replace in that construction the segments $[2^{-1},3]$ by the segments $[2^{-2},3]$. The figure 2 shows the set $S(0) \cup S(1) \cup S(2)$.

Formal description of $S(n)$, $n > 1$, is as follows. Put $C(n,i) = 3^{-n} \ast C(n-1,i)$ if $i = 1,2,\cdots,2^n$, and $c(n,i) = \frac{2}{3} + 3^{-n} \ast C(n-1,i-2^n)$ if $i = 2^n+1,\cdots,2^{n+1}$. Let $S(n,i) = \cup \{\{c\} \times [2^{-n},3]: c \in C(n,i)\}$, where $i = 1,2,\cdots,2^{n+1}$. Let $u(n,i) = 3^{-n} \ast u(n-1,i)$ if $i = 1,2,\cdots,2^n$, and $u(n,i) = \frac{2}{3} + 3^{-n} u(n-1,i)$ if $i = 2^n+1,\cdots,2^{n+1}$. Let $S(n) = \cup \{S(n,i): i = 1,2,\cdots,2^{n+1}\} \cup S^+(u(n,1),3); C(n,1) \times \{3\} \cup S^-((u(n,2),2^{-n}); C(n,1) \times \{2^{-n}\}) \cup S^+((u(n,3),2^{-n}); C(n,4) \times \{3\}) \cup \cdots \cup S^+((u(n,2^{n+1}),3); C(n,2^{n+1}) \times \{3\})$.

Let $X = \cup \{S(n): n = 0,1,2,\cdots\}$. Observe that any quasi-component $K(c)$ of $X$ is the union of a segment-like set $\{c\} \times I(c)$ and sin$(\frac{1}{x})$-like curve emerging from $(\frac{4}{9} + \frac{1}{9} c,3)$, where $c \in C$. By the theorem of Katsuura the quotient $Y = X/C \times \{0\}$ is a connected space and $q(C \times \{0\})$
is the dispersion point. By \( q \) we denote the quotient map from \( X \) onto \( Y \).

Let \( g \) be a linear and order-preserving mapping from \( C(n,i) \) onto \([0,3^{-n}] \cap C \) if \( i \equiv 2 \) (mod 4), and onto \([\frac{2}{3^n}, \frac{3}{3^n}] \cap C \) if \( i \equiv 3 \) (mod 4), and let \( g \) be a linear and order-reversing mapping from \( C(n,i) \) onto \([0,3^{-n}] \cap C \) if \( i \equiv 1 \) (mod 4), and onto \([\frac{2}{3^n}, \frac{3}{3^n}] \cap C \) if \( i \equiv 0 \) (mod 4). Let the map \( f \) from \( X \) into itself be defined as follows:

\[
\begin{align*}
f(x) &= (0,1) \text{ if } x \in S(0), \\
f(a,b) &= (0,0) \text{ if } b \geq \frac{5}{2}, \\
f(a,b) &= (g(a), \frac{5}{2} - b) \text{ if } \frac{3}{2} < b < \frac{5}{2}, \\
f(a,b) &= (g(a),1) \text{ if } (a,b) \in S(n,i) \text{ and } b \leq \frac{3}{2}, \\
f(x) &= (g(c),1) \text{ if } x \in S^-(u(n,i), 2^{-n}); (c,2^{-n})) \\
&\text{ for some } c \text{ in } C(n,i).
\end{align*}
\]

Let \( f_q \) denote a map from \( Y \) into itself induced by \( f \). The map \( f_q \) is a continuous and non-constant function, and the dispersion point is not a fixed point of the map. The proof of continuity is straightforward but tedious.

**Example 2.** We modify the example 1 to obtain a mapping onto. Let \( f \) and \( X \) have the same meaning as in the example 1. For any point \( c \) in the Cantor set \( C \) let \( D(c) \) denote the set of all the points on the segment joining \((\frac{4}{9} + \frac{1}{9} \cdot c,4)\) and \((c,5)\) the second coordinate of which is rational if \( c \) is rational, and irrational if \( c \) is likewise. Let

\[
\begin{align*}
X(0) &= X \cup U\{D(c) : c \in C\} \cup U\{[c] \times [3,4](c) : \\
&\quad c \in C(1,2) \cup C(1,3)\} \quad \text{(see figure 3)}.
\end{align*}
\]

Let \( X(n) = (0,5) + X(n-1) \) for \( n = 1,2,3,\ldots \). Put \( X(\infty) = U\{X(n) : n = 0,1,2,\ldots\} \). Let \( F \) be a mapping from \( X(\infty) \)
into itself defined by

\[ F|X = f \]

\[ F(x) = (0,0) \text{ if } x \in X(0) \setminus X \]

\[ F(x) = x - (0,5) \text{ if } x \in X(n), \ n = 1,2,3,\ldots \]

It is easy to see that \( F \) is onto.

Let \( Z \) be the quotient space \( X(\infty)/C \times \{0\} \), let \( q \) be the quotient map from \( X(\infty) \) onto \( Z \) and let \( F_q \) be the function on \( Z \) induced by \( F \). Observe that \( Z \) is a connected subset of the plane with the dispersion point \( q(C \times \{0\}) \), \( F_q \) is a continuous function from \( Z \) onto itself, and the dispersion point is not a fixed point of \( F_q \).

References


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