A FINITE TO ONE OPEN MAPPING PRESERVES SPAN ZERO

by

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It is shown that span zero is preserved under finite
to one open mappings. 54C10, 54F20

Introduction

The notion of span of a compact metric space and its
natural generalization semispan were defined by A. Lelek
in 1964 and 1977, respectively [6, 7]. It follows from
the definitions that semispan is greater than or equal to
span and both functions are monotone with respect to
closed subsets. It can be shown directly that a nonuni­
coherent continuum or a triod have span greater than zero.
All chainable continua have semispan zero and those con­
tinua without the fixed point property have span greater
than zero. J. F. Davis has shown that span zero and semi­
span zero are equivalent [2]. In [11] I. Rosenholtz has
shown that an open image of a chainable continuum is also
chainable. A natural question is whether open mappings
preserve span zero.

Notations and Definitions

A metric continuum is a compact connected metric
space. We let d represent the distance function and \( \Pi_1 \)
and \( \Pi_2 \) the natural projection mappings of the Cartesian
Product \( X \times X \) onto \( X \). The semispan of a compact metric
space, denoted by \( \sigma_0(X) \), is the least upper bound of all real numbers \( \varepsilon \) such that there is a subcontinuum \( Z \) of \( X \times X \) with the properties that \( \Pi_2(Z) \subseteq \Pi_1(Z) \) and \( d(x,y) > \varepsilon \) for all \((x,y) \in Z\). The definition of span of \( X \), denoted by \( \sigma(X) \), is the same except that we require \( \Pi_2(Z) = \Pi_1(Z) \).

A mapping \( f(X) = Y \) is weakly confluent provided that for each subcontinuum \( K \subseteq Y \), there is a component \( H \) of \( f^{-1}(K) \) such that \( f(H) = K \). A continuum \( Y \) is in class \( W \) provided that for each mapping \( f \) from a continuum onto \( Y \), \( f \) is weakly confluent. A continuum is indecomposable provided it cannot be the union of two proper subcontinua. For \( x \in X \), define \( K_x \) as follows; \( K_x = \cap \{ K, K \text{ is a subcontinuum of } X \text{ and } x \text{ is in the interior of } K \} \). The properties of \( K_x \) for certain continua are developed in [1]. A subcontinuum \( B \) is terminal in the space \( X \) if whenever \( H \) and \( K \) are two subcontinua with \( B \cap H \neq \emptyset \) and \( B \cap K \neq \emptyset \), then \( B \cup H \subseteq B \cup K \) or \( B \cup H \supset B \cup K \).

**Preliminary Results**

A mapping is finite to one if each point inverse is a finite set.

**Lemma 1.** Let \( f(X) = Y \) be a finite to one open mapping, where \( X \) and \( Y \) are locally compact separable metric spaces. The set \( D \) of all points \( x \) of \( X \) such that \( f \) is a local homeomorphism at each point of \( f^{-1}f(x) \) is an open dense subset of \( X \).
Proof. Let \( D_n = \{ x | x \in X \text{ and } f^{-1}(f(x)) \text{ has less than or equal to } n \text{ points} \} \). Each \( D_n \) is a closed set by the openness of \( f \). Since \( X = \bigcup_{n=1}^{\infty} D_n \) and \( X \) is a Baire Space some \( D_n \) contains interior points. Let \( n_0 \) be the least integer for which \( D_{n_0} \) has interior points and let \( x \) be an interior point. Each point of \( f^{-1}(f(x)) \) is interior to \( D_{n_0} \) since \( f^{-1}(\text{int} M_{n_0}) \) is an open set. Let \( f^{-1}(f(x)) = \{ x_1, x_2, \ldots, x_{n_0} \} \) and take pairwise disjoint open sets \( U(x_i) \), \( i = 1, 2, \ldots, n_0 \) about each point of \( f^{-1}(f(x)) \) and contained in \( \text{int} M_{n_0} \). Now \( f \mid U(x_i) \), \( i = 1, 2, \ldots, n_0 \) is an open and 1-1 mapping of \( U(x_i) \) onto \( f(U(x_i)) \) and is therefore a homeomorphism. To show \( D \) is dense in \( X \) apply the argument above to an arbitrary non-empty open set \( U \). That is \( U = \bigcup_{n=1}^{\infty} (U \cap D_n) \) and \( U \) as a subspace is a Baire space so there is a least integer \( n_0 \) such that \( U \cap D_{n_0} \) has interior points. See [12], VII, 3.5 for a similar result.

Lemma 2. If \( f(X) = Y \) is a finite to one open mapping, where \( X \) and \( Y \) are locally compact separable metric spaces, and \( B \) is a closed subset of \( X \) such that \( \text{int} f(B) \) is not empty, then \( \text{int} B \) is not empty.

Proof. By the preceding result there is a point \( y_0 \in \text{int} f(B) \) such that \( f \) is a local homeomorphism at
each point of $f^{-1}(y_0)$. Suppose $f^{-1}(y_0) = \{x_1, \ldots, x_{n_0}\}$

and choose pairwise disjoint open sets $U(x_i), i = 1, \ldots, n_0$

so that $W = f(U(x_i)) = f(U(x_j)) \subset \text{int. } f(B)$ for all

$i, j = 1, \ldots, n_0$ and each $f|U(x_i)$ is a homeomorphism of

$U(x_i)$ onto $f(U(x_i))$. We have $W = \bigcup_{i=1}^{n_0} f(B \cap U(x_i))$ and each

$f(B \cap U(x_i))$ is an $F_0$ set and $W$ as a subspace is a Baire

space so some $f(B \cap U(x_i))$ has interior points. If $V$ is

an open set in $f(B \cap U(x_i))$ then the set $U$ in $B \cap U(x_i)$

which maps onto $V$ is open in $X$.

Lemma 3. Let $X$ be a hereditarily unicoherent atroidic

metric continuum. For $x \in X$, if $K_x$ has non-empty interior

then $K_x$ is indecomposable or the union of two indecompos­

able continua with non-empty interiors.

Proof. Suppose $K_x = L \cup M$, where $L$ and $M$ are proper

subcontinua. We can assume $L = L - M$ and $M = M - L$. By

the definition of $K_x$ it follows that $x \not\in L - M$ and

$x \not\in M - L$ so $x \in L \cap M$ and furthermore $x \in \text{int.} (L \cup M)$

otherwise the space $X$ contains a triod. Suppose $L = L_1 \cup L_2$, where $L_1$ and $L_2$ are proper subcontinua. If

$x \not\in L_1$, then $x \in \text{int.} (L_2 \cup M)$ which contradicts the defini­

tion of $K_x$ so that $x \in L_1 \cap L_2 \cap M_1$ this implies $L_1 \cup L_2 \cup

M$ is a triod and this is not possible hence $L$ and $M$ are

indecomposable with non-empty interiors.

From [3] we have: Theorem 2. If $\sigma_0(X) = \varepsilon > 0$,

there exists an indecomposable continuum $I \subset X$ with
\[ \sigma_0(I) = \varepsilon \] and every proper subcontinuum of \( I \) has semispan less than \( \varepsilon \). The space \( X \) in this result must be an atroidic hereditarily unicoherent metric continuum.

**Main Result**

The following theorem is the main result of this article.

**Theorem.** Let \( X \) and \( Y \) represent metric continua.

If \( \sigma_0(X) = 0 \) and \( f(X) = Y \) is a finite to one open mapping, then \( \sigma_0(Y) = 0 \).

**Proof.** \( \sigma_0(X) = 0 \) implies \( X \) is atroidic and hereditarily unicoherent. By [8] \( \sigma_0(X) = 0 \) implies \( X \) is tree-like and by [5] atroidic and tree-like implies \( X \) is in class \( W \). By [9] \( Y \) is hereditarily unicoherent and by [10] \( Y \) is tree-like. The mapping \( f \) takes atroidic continua onto atroidic continua, so by [5], \( Y \) is in class \( W \). By Theorem 2, if \( \sigma_0(Y) = \varepsilon > 0 \), then there exists an indecomposable subcontinuum \( I \) with \( \sigma_0(I) = \varepsilon \) and every proper subcontinuum of \( I \) has semispan less than \( \varepsilon \). Let \( H \) be a component of \( f^{-1}(I) \), then, as is well known, \( f(H) = I \) and \( f/H \) is a finite to one open mapping of \( H \) onto \( I \). For \( x \in H \), \( K_x = \bigcap_{a \in \Gamma} K^a_x \), \( K_x \subseteq H \), \( a \in \Gamma \) and \( x \in \text{int} \ K^a_x \) relative to \( H \). The continuum \( f(K^a_x) \subseteq I \) has interior points so \( f(K^a_x) = I \) since \( I \) is indecomposable. Thus, in this case, we can argue that \( f(K_x) = I \) and since \( f \) is finite to one and open, by Lemma 2 \( \text{int} \ K_x \neq \emptyset \). There are only two possible cases; (i) \( K_x \) is indecomposable, or (ii) \( K_x = K_y \cup K_z \), where \( K_y \) and \( K_z \) have non-empty interior, are indecomposable,
\[ \text{int} \ K_y \cap \text{int} \ K_z = \emptyset \text{ and } x \in K_y \cap K_z. \] This means that, since \( f \) is finite to one and open on \( H \), we can consider \( H \) as a finite linear chain of indecomposable continua, i.e., \( H = K_{x_1} \cup K_{x_2} \cup \ldots \cup K_{x_n}, \) where \{int \( K_{x_i} \)\}, \( i = 1, \ldots, n \) is a pairwise disjoint collection and only successive \( K_{x_i} \)'s intersect. The continuum \( H \) is irreducible from a point \( p \in K_{x_1} \) to a point \( q \in K_{x_n}. \) There is a continuum \( Z \subseteq I \times I \) such that for all \((x,y) \in Z, d(x,y) \geq \varepsilon \) and \( \Pi_1(Z) = I = \Pi_2(Z). \) The continuum \( Z \) can be chosen so that it is indecomposable and \( \Pi_1/Z \) and \( \Pi_2/Z \) are irreducible mappings [4]. Let \( C \) be the composant of \( K_{x_1} \) which is accessible from \( K_{x_1} \) and let \( L \) be the composant of \( I \) which contains \( f(C). \) The composant \( L \) may be expressed as, \( L = \bigcup_{i=1}^{\infty} D_i, D_i \subseteq D_{i+1}, i = 1, 2, \ldots \) and each \( D_i \) is a continuum. For each \( i, Z - \Pi_1^{-1}(D_i) \) is open in \( Z \) and connected and dense since \( \Pi_1^{-1}(D_i) \) cannot meet all composants of \( Z. \) Thus \( \bigcap_{i=1}^{\infty} Z - \Pi_1^{-1}(D_i) \cap \bigcap_{i=1}^{\infty} Z - \Pi_2^{-1}(D_i) \neq \emptyset \) by a Baire theorem and hence there exists an \((x,y) \in Z \) with \( x, y \in I - L. \) Since \((f \times f)(K_{x_1} \times K_{x_1}) = I \times I \) there is a point \((a,b) \in K_{x_1} \times K_{x_1} \) with \((f \times f)(a,b) = (x,y) \) and \( a, b \in K_{x_1} - C. \) Let \( B \) be the component of \((f \times f)^{-1}(Z) \) which contains \((a,b). \) We have \( f(\Pi_1(B)) = I \) so \( \Pi_1(B) \) has
interior points and thus cannot be a proper subcontinuum of $K_{x_1}$. The point $a \in \Pi_1(B)$ and $a \not\in C$ so $\Pi_1(B) \supset K_{x_1}$.

Similarly $\Pi_2(B) \supset K_{x_1}$. Since $K_{x_1}$ is a terminal subcontinuum in $H$ it follows that $\Pi_1(B) \supset \Pi_2(B)$ or $\Pi_1(B) \subset \Pi_2(B)$. Since $B$ does not meet the diagonal of $H$, we have $\sigma_0(H) > 0$ and consequently $\sigma_0(X) > 0$, which is a contradiction.

References


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