HOMEOMORPHISMS OF A SOLID HANDLEBODY AND HEEGAARD SPLITTINGS OF THE 3 SPHERE $S^3$

by

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Let $H_g$ be a solid handlebody of genus $g$, with boundary $\partial H_g = F_g$. Let $B = \{a_1, b_1, a_2, b_2, \ldots, a_g, b_g\}$ be a fixed system of basecurves based at a common basepoint $0$, such that the $a_i$'s are meridian circles of $H_g$. Let $M_g$ denote the mapping class group of the closed orientable surface $F_g$. And let $K_g$ denote the subgroup of $M_g$ consisting of mapping classes induced by some homeomorphism of the handlebody $H_g$. An element of $K_g$ will be called an extendible mapping class.

The subgroup $K_g$ plays a very important role in Heegaard splitting of 3-manifolds (Cf. [1] & [8]). In this paper, we describe this subgroup explicitly by giving a finite set of generators in the first two sections. Comparing to Suzuki's generators [7], not only is the number of generators one less, but also the expressions in the generators of the mapping class group $M_g$ are quite easy. In the third section, all Heegaard splittings of the 3-sphere $S^3$ are explicitly given, this was asked in Hempel's book ([3] p. 164).

1. Some Extendible Mapping Classes

First we are going to give some extendible mapping classes, show they generate the group $K_g$, then reduce
the number by using the technique given in the papers [4] and [5].

Recall that the mapping class group $\mathcal{M}_g$ is generated by three elements: the linear cutting $L$, the normal cutting $N$ and the transport $T$. Algebraically, they are given by

$$L = [a_1, b_1, a_2, b_2, \ldots, a_g, b_g],$$
$$N = [x_2 a_2, b_1, a_1, x_2 b_2, a_2, a_3, b_3, \ldots, a_g, b_g],$$
where $x = [a_1, b_1][a_2, b_2]$, and

$$T = [a_g, b_g, a_1, b_1, \ldots, a_{g-1}, b_{g-1}].$$

We also denote by $M = NLM$ the meridian cutting, $P = LML$ the parallel cutting, $Q = TPT = N^3PN^3$ the parallel cutting of the second handle, $c_i = [a_i, b_i]$ the waist of the $i$-th handle, and $x = c_1 c_2$ the waist of the first two handles.

Now we list some elementary extendible mapping classes.

1) The meridian cutting $M$, given by

$$M = [a_1, b_1, a_2, b_2, \ldots, a_g, b_g].$$

2) The transport $T$.

3) The handle rotation $\phi$, (Figure 1.1), given by

$$\phi = [c_1, \bar{a}_1, \bar{b}_1, \bar{a}_1, \bar{b}_1, a_2, b_2, \ldots, a_g, b_g],$$

is obtained by a $180^\circ$-rotation of the first handle along its waist circle $c_1$. 

4) The handle switching $\psi$, (Figure 1.2), given by

$$\psi = [c_1a_2\overline{c}_1, c_1b_2\overline{c}_1, a_1, b_1, a_3, b_3, ...],$$

is obtained by moving the second handle around the first handle into the position in front of the first one.

5) The handle rounding $\sigma$, (Figure 1.3), given by

$$\sigma = [a_1, b_1\overline{a}_1\overline{b}_1c_2b_1a_1, \overline{a}_1c_1a_2\overline{c}_1a_1, \overline{a}_1c_1b_2\overline{c}_1a_1, a_3, b_3, ...],$$

is obtained by moving one foot of the first handle around the second one.
Figure 1.3 Handle rounding $\sigma$

Figure 1.4 Handle crossing $\chi$

Figure 1.5 One-foot sliding $\omega$
6) The handle crossing $\chi$, (Figure 1.4), given by

$$\chi = [c_1a_2,b_2,b_2a_1b_2,b_2b_1b_2,a_3,b_3,\ldots],$$

is obtained by sliding the whole first handle along the longitude circle $b_2$ of the second handle.

7) The one-foot sliding $\omega$, (Figure 1.5), given by

$$\omega = [a_1,b_1,b_2c_1a_1\bar{a}_2a_1b_2a_1b_1b_2,b_2c_1a_1b_2a_1c_1b_2,a_3,b_3,\ldots],$$

is obtained by sliding one foot of the first handle along the longitude circle $b_2$ of the second handle.

8) The one-foot knotting $\theta$, (Figure 1.6), given by

$$\theta = [a_1,\bar{a}_2b_1,\bar{a}_2c_1a_1\bar{a}_2a_1b_2a_1c_1b_2,a_3,b_3,\ldots],$$

is obtained by moving one foot of the first handle along the meridian circle $a_2$ of the second handle.

9) The handle replacing $\eta$, (Figure 1.7), given by

$$\eta = [\bar{a}_1c_1a_2,\bar{a}_2\bar{c}_1b_1a_2,\bar{a}_2c_1b_1a_2b_1c_1a_2,b_2b_1c_1a_2,a_3,b_3,\ldots],$$

is obtained by replacing the first handle with the cylinder between the first and the second handles.
Remark. By definition, among all elementary extendible mapping classes, the operations $T$, $\phi$, $\psi$, $\sigma$, $\chi$, $\omega$ and $\eta$ can be obtained by an isotopy deformation of $S^3$ (i.e., obtained by moving the handlebody $B_g$ inside of $S^3$ without cutting it open). And the operation $\theta$ is a combination of $\omega$ and meridian twists, which can be obtained in the following way: pass the left foot of the first handle along the longitude $B_2$ of the second handle in the anti-clockwise way, twist the second handle along its meridian,
pull back the foot of the first handle along the new longitude, and adjust the longitude $b_1$ by a twist along the meridian of the first handle. Precisely, 
\[ \theta = \omega \cdot TMT \cdot \bar{\omega} \cdot M. \]

**Theorem 1.1.** In the mapping class group $\mathcal{M}_g$, we have the following expressions:

(i) $\phi = p^2 = (LM)^4$,  
(ii) $\psi = \bar{P}^4N^3 = N^3Q^4$,  
(iii) $\sigma = \bar{P}(LN)^5(LN)^5\bar{P}Q^4 = (\bar{P}QN)^2Q^4$,  
(iv) $\chi = \bar{L}(NL)^5(NL)^5 = (QNP)^2N^3$,  
(v) $\omega = Q^3NQ^2P$,  
(vi) $\theta = Q^2NQP = \bar{Q}\omega Q$,  
(vii) $\eta = \bar{P}Q^{-2}NQ = P\omega P$.

**Proof.** The expressions are found by using the algorithm given in [4] and [5], which certainly was not easy. After the formulas have been discovered, the proof is just an immediate verification.

For example, for (iii), we know that,
\[ (LN)^5 = [x\bar{a}_1, \bar{B}_1, x, \bar{a}_1, \bar{B}_1, c_2, \bar{a}_2, b_1, a_1, \bar{a}_1, \bar{B}_1, \bar{B}_2, \bar{c}_2, b_1, a_1, \]
\[ a_3, b_3, \ldots], \]

thus \[ (NL)^5 = [\bar{a}_1, x, \bar{B}_1, x, \bar{a}_1, \bar{B}_1, c_2, \bar{a}_2, b_1, a_1, x, \bar{a}_1, \bar{B}_1, \bar{B}_2, \bar{c}_2, b_1, a_1, x, \]
\[ a_3, b_3, \ldots], \]

and \[ (L\bar{N})^5 = [\bar{a}_1, c_1, x, \bar{B}_1, x, \bar{a}_1, \bar{a}_2, c_1, a_1, x, \bar{a}_1, c_1, \bar{B}_2, \bar{x}a_1, x, \]
\[ a_3, b_3, \ldots], \]

then \[ (LN)^5(L\bar{N})^5 = [x\bar{c}_1, a_1, b_1, b_1, x, \bar{a}_2, \bar{B}_1, b_1, \bar{b}_2, \bar{x}B_1, \]
\[ a_3, b_3, \ldots]. \]
Because \( P = [c_1 b_1, \bar{a}_1, a_2, b_2, \ldots], \)
and \( Q^4 = [a_1, b_1, c_2 a_2 c_2, \bar{c}_2 b_2 c_2, a_3, b_3, \ldots], \)
\[
\overline{P}(LN)^5 (LN)^5 P Q^4 = \overline{P}[x b_1, \bar{a}_1, a_1 c_1 a_2 \bar{c}_1 a_1, \bar{a}_1 c_1 b_2 \bar{c}_1 a_1, a_3, b_3, \ldots] = \sigma.
\]

Also we have,
\[
\overline{P} Q N = [a_1, \bar{a}_1 x b_1 a_1, a_1 c_1 a_2 c_1 a_1, \bar{a}_1 c_1 b_2 c_1 a_1, a_3, b_3, \ldots]
\]
and \( (\overline{P} Q N)^2 = [a_1, \bar{a}_1 x b_1 a_1, a_1 x a_2 x a_1, \bar{a}_1 x b_2 x a_1, a_3, b_3, \ldots], \)

Clearly, \( \sigma = Q^4 (\overline{P} Q N)^2 = (\overline{P} Q N)^2 Q^4. \) And similarly we can prove the other formulas.

2. Generators of the Subgroup \( K_g \)

In this section, we are going to prove that,

**Theorem 2.1.** The extendible mapping class subgroup \( K_g \) of the surface mapping class group \( M_g \) is generated by five elements:

\[
M, T, N^3, P^2, \text{ and } PN^2 P,
\]

and also by the five elements:

\[
T, N^3, \overline{N} L N, N L N^2 \overline{L}, \text{ and } L N L^2 N L.
\]

Regard the handlebody \( H_g \) as the down-semispace of the Euclidean space \( E^3 \) with \( g \) pairs of holes on its boundary \( F_g \) identified (Figure 2.1). Instead of the basecurves \( B = \{a_i, b_i\}_{1 \leq i \leq g} \), we will study the basearcs \( B = \{p_i, q_i, r_i\}_{1 \leq i \leq g} \), where as joining the oriented arcs, we have
The disk holes, denoted by $D_i$'s, will be chosen as the meridian disks, which form a cutting system of the handlebody $H_g$. Their boundary circles, $r_i$'s, are the fixed meridian circles. In the plane in Figure 2.1, the disks $D_i$ and the circles $r_i$ are split in two. We will denote by $D_i'$ and $D_i''$ the two copies of $D_i$, denote by $r_i' = \partial D_i'$ and $r_i'' = \partial D_i''$ the two copies of $r_i$, and call them the cutting disks and cutting circles respectively. Moreover, we also suppose that $r_i'$ contains an endpoint of $p_i$ and $r_i''$ contains one of $q_i$.

We call this new description the planar representation of $F_g$. Using it, a mapping class of the surface $F_g$ may be drawn easily in the plane. For example, the mapping classes $\phi$, $\psi$ and $\theta$ are drawn in Figure 2.2, and it is quite easy to understand how they move the feet of handles.
Using $\phi$, $\psi$, $\theta$ and $T$, we construct some more elementary movings of handle-feet. A family of mapping classes, called the $i$-th foot knotting $\Theta_i$ and the $i$-th foot knotting $\Theta'_i$, is defined by moving the foot $r_1^i$ of the first handle along the meridian circle $a_i$ and the meridian circle $b_i\overline{a}_i\overline{B}_i$, i.e. $r_1^i$ and $r_1^{i'}$, respectively, (see Figure 2.3). Therefore, $\Theta'_i = \phi_i \Theta_i \overline{\Theta}_i$, where $\phi_i = T^{i-1} \phi T^{i-1}$, for $i = 2, \ldots, g$. Precisely, we have
Proposition 2.2. The $i$-th foot knotting and $I$-th foot knotting are generated by the mapping classes $T$, $\psi$, $\psi$, and $\theta$. Furthermore, they have the following expressions:

$$\Theta_i = T_i^{-2}(\psi T)^{i-2} \theta(T\psi)^{i-2} T_i^{-2},$$

and

$$\Theta'_i = T_i^{-1} \psi T(\psi T)^{i-2} \theta(T\psi)^{i-2} T_i^{1-i}.$$

Proof. By Figure 2.3,

$$\Theta_i = [a_1, c_2, \ldots c_{i-1}, a_i, \bar{c}_i, \ldots \bar{c}_{2}, a_{i-1}, \ldots]$$

$$= [a_1, c_2, \ldots c_{i-1}, a_i, \bar{c}_i, \ldots \bar{c}_{2}, a_{i-1}, \ldots]$$

$$\Theta'_i = [a_1, c_2, \ldots c_{i-1}, a_i, \bar{c}_i, \ldots \bar{c}_{2}, a_{i-1}, \ldots]$$

$$= [a_1, c_2, \ldots c_{i-1}, a_i, \bar{c}_i, \ldots \bar{c}_{2}, a_{i-1}, \ldots]$$
Then, a direct calculation implies the proposition. For example, let us compute $\psi_i$. Let $\psi_j = T_{j-1}^j \psi^{j-1}$ and $z = c_2 c_3 \cdots c_{i-1}$, then

$$\psi_2 \cdot \psi_{i-1} = [a_1, b_1, z_1, z_2, a_2, b_2, \ldots, a_{i-1}, b_{i-1}, a_{i+1}, b_{i+1}, \ldots],$$

and

$$\psi_{i-1} \cdot \psi_2 \cdot \psi_{i-1} = [a_1, a_1, z_1, z_2, a_2, b_2, \ldots, a_{i-1}, b_{i-1}, a_{i+1}, b_{i+1}, \ldots].$$

Now we want to start proving that the elementary extendible mapping classes generate the group $K_g$.

Let $f$ be an extendible mapping class, i.e. an element of $K_g$. The idea is to find another extendible mapping class $g$ generated by our generators, such that either $gf$ or $fg$ becomes "simpler" than $f$. The process will be repeated until the identity map is obtained.

**Lemma 2.3.** Let $a$ be an oriented simple arc on the surface $F_g$ from the basepoint $Q$ to the endpoint $Q$ of $q_1$ at $r_i$, which does not intersect any of the meridian circles $r_i$ for all $i$, and does not intersect any of the arcs $p_j$ and $q_j$ for $j \geq s$. Then, there exists a self-homeomorphism $g$ whose homeotopy class is generated by the classes $T$, $M$, $\phi$, $\psi$, and $\theta$, such that $(q_1)g = a$, $(r_i)g = r_i$, for any $i \geq 1$. 
Furthermore, \((p_j)g = p_j\) if \(a \cap p_j = \{0\}\), and \((q_j)g = q_j\) if \(a \cap q_j = \{0\}\), for any \(j \geq 1\).

**Proof.** Suppose the arc \(a\) intersects \(q_1\) transversally, and denote by \(k\) the number of points of the intersection \(a \cap q_1\) other than 0 and \(Q\). When \(k = 0\), the union of these two curves becomes a simple closed curve \(\gamma = \overline{aq_1}\).

![Diagram of curves and arcs](image)

**Figure 2.4**

If \(\gamma\) does not intersect any of the arcs \(p_i\) and \(q_i\) other than \(q_1\), we may let \(g\) either be an isotopy if \(\gamma\) does not separate the circles \(r_1^l\) and \(r_1^r\), or a meridian twist from \(M_1^+\) if the disk area bounded by \(\gamma\) includes \(r_1^r\) (Figure 2.4). Otherwise, let \(P\) be an intersection point closest to \(Q\) along \(a\). If \(P \in p_1\) we may use the mapping class \(\phi\) to remove it, and if \(P \in p_i\) or \(q_i\), for some \(i \geq 2\), we may use the mapping class \(\Theta_i\) or \(\Theta_i\) given in Proposition 2.2 to remove it. Actually, \(g\) will be the mapping which moves the cutting circle \(r_1^r\) along the curve \(\gamma\), its explicit expression in mappings \(\Theta_i\)'s, \(\Theta_i\)'s and \(\phi\) may be easily found from the intersection set \(\gamma \cap (U(p_1 \cup q_1))\) along the curve \(\gamma\). This clearly leaves the unintersected \(p_1\)' and \(q_i\)'s unchanged.
Suppose $k \geq 1$, and let $P$ be the intersection point of $\alpha$ and $q_1$ closest to the point $0$ along $\alpha$. Let $\beta = \alpha |_{OP} q_1 |_{PQ}$. After an isotopy deformation, we have the intersection numbers $k(q_1, \beta) = 0$ and $k(\beta, \alpha) \leq k - 1$. And clearly $\beta$ does not intersect other $p_i$'s and $q_i$'s more than $\alpha$ does, since we have

$$\beta \cap (U(p_i \cup q_i)) = \alpha |_{OP} \cap (U(p_i \cup q_i)) \subseteq \alpha \cap (U(p_i \cup q_i)),$$

(Figure 2.5). By induction, we have $g_1$ and $g_2$ generated by $T$, $M$, $\phi$, $\psi$, $\theta$ and $\eta$, such that $(q_1)g_1 = \beta$ and $(\beta)g_2 = \alpha$. Then, take $g = g_1 g_2$.

**Lemma 2.4.** Let $f$ be an arbitrary self-homeomorphism of the handlebody $H_g$, such that $(r_i)f = r_i$, for all $i$. Then, the homeotopy class of $f$ is generated by the classes $T$, $M$, $\phi$, $\psi$, and $\theta$.

**Proof.** This is a direct consequence of Proposition 2.2 and Lemma 2.3. Inductively, suppose we have $(p_i)f = p_i$ and $(q_i)f = q_i$ for $i \leq s - 1$, for some $s$. 
Rotate the handles until \((p_s, q_s)\) is in the first position, switch \(p_s\) and \(q_s\) by \(\phi\), simplify \(p_s\) by using Lemma 2.3, then switch back to simplify \(q_s\) in the same way, and finally rotate it back. Again by Lemma 2.3, all \(p_i\)'s and \(q_i\)'s for \(i < s - 1\) are unchanged.

![Diagram](image)

**Figure 2.6**

\[(r_2)x = r_1, \ (r_i)x = r_i \text{ for } i \geq 3\]

By Lemma 2.4, from now on, it is sufficient to study the image of the cutting system \(r_i\)'s of an extendible class. Thus, we first discuss some extendible mapping classes which change the cutting system. For example, the images of the cutting system of the mapping classes \(\chi, \omega\) and \(\eta\) are drawn in Figures 2.6-8.

**Lemma 2.5.** Let \(\gamma\) be an oriented simple closed curve on the surface \(F_g\) which does not intersect any of the meridian circles \(r_i\), and whose homology class in \(H_1(F_g, \mathbb{Z})\) relative to the meridian circle \(r_i\) is nontrivial (i.e., \(\gamma\) separates \(r_i\) and \(r_i''\) in two sides in the planar representation). Then, there exists a self-homeomorphism \(g\) whose homeotopy class is generated by the classes \(T, M, \phi, \psi, \theta\) and \(\eta\), such that \((\gamma)g = r_1\) and \((r_i)g = r_i\), for any \(i \geq 2\).
Proof. Denote by $k$ the number of cutting circles in the disk area $\Delta$ bounded by $\gamma$ in the planar representation of $F_g$. The lemma will be proved by induction on $k$.

For $k = 1$, the cutting circle in $\Delta$ must be either $r_1'$ or $r_1''$. If $\gamma$ is oriented in the same way as this cutting circle, we may let $g$ be an isotopy deformation, which deforms $\gamma$ into $r_1$. If $\gamma$ is oriented in the opposite way, follow the isotopy by the operation $\phi$, which reverses the orientation of $r_1$.

For $k = 2$, by some handle switchings and rotations, i.e. a mapping class generated by $\phi$, $\psi$ and $T$, we may
suppose that the two cutting circles in $\Delta$ are $r_1$ and $r_2$. Connecting the point $P = p_2 \cap r_2$ and the point $Q = q_1 \cap r_1$ by a simple arc $\delta$ in $\Delta$ which intersects neither $q_1$ nor $p_2$ (Figure 2.9).

If $\delta$ does not intersect any other $p_i$'s and $q_i$'s, the disk $\Delta$ is isotopic to a neighborhood of $r_1 \cup \delta \cup r_2$ whose boundary is exactly the circle $(r_1)\cap$ as shown in Figure 2.9. Thus, the lemma is done.

If $\delta$ does intersect some $p_i$'s or $q_i$'s, we may simplify the intersection by the method we did in Lemma 2.3. Actually, letting $\alpha = p_2 \delta$, apply Lemma 2.3 to reduce to the previous case.

For $k \geq 3$, by some handle switchings and rotations, i.e. a mapping class generated by $\phi$, $\psi$ and $T$, we may suppose again that the cutting circles $r_1$ and $r_2$ are in the
domain $\Delta$. Connecting the point $P = p_2 \cap r_2'$ and the point $Q = q_1 \cap r_1''$ by a simple arc $\delta$ in $\Delta$ which intersects neither $q_1$ nor $p_2$ (Figure 2.10), we may choose a disk neighborhood $\Delta'$ of $r_1'' \cup \delta r_2'$ contained in the interior of $\Delta$ but including no other cutting circles. Applying the case of $k = 2$ to the disk $\Delta'$, the original $\Delta$ will be reduced to the case of $k - 1$.

Applying Lemma 2.5 repeatedly, we have the following immediate consequence.
Lemma 2.6. Let $f$ be an arbitrary self-homeomorphism of the handlebody $H_g$, with the property that,

$$(r_i)f \cap r_j = \emptyset, \text{ for all } i, j = 1, 2, \ldots, g.$$  

Then, there exists another self-homeomorphism $g$ whose homeotopy class is generated by the classes $T, \phi, \psi, \theta$ and $\eta$, such that

$$(r_i)g = (r_i)f, \text{ for } i = 1, 2, \ldots, g.$$  

Lemma 2.7. Let $f$ be an arbitrary self-homeomorphism of the handlebody $H_g$, then there exists another self-homeomorphism $g$ whose homeotopy class is generated by the classes $T, \phi, \psi, \theta$ and $\eta$, such that

$$\bigcup_{i=1}^{g} (r_i)f \cap \bigcup_{i=1}^{g} (r_i)g = \emptyset.$$  

i.e. none of the circles $(r_i)g$'s intersects a meridian circle of $r_i$'s.

Proof. Denote by $k_i$, for $i = 1, 2, \ldots, g$, and $k$ the numbers of intersection points given by

$$k_i = \#((r_i)f \cap \bigcup_{j=1}^{g} r_j) \text{ and}$$

$$k = \bigoplus_{i=1}^{g} k_i = \#(\bigcup_{i=1}^{g} r_i)g \cap (\bigcup_{j=1}^{g} r_j).$$

For $k = 0$, take $g$ to be the identity.

For $k > 1$, we may suppose $k_1 \neq 0$, i.e. $(r_1)f \cap (\bigcup_j r_j) \neq 0$. Consider the meridian disks $D_1$ bounded by the $r_1$ in the solid handlebody $H_g$, which have nonempty intersection with the disk $(D_1)f$. By an isotopy deformation, we can suppose the set $(D_1)f \cap (\bigcup_j D_j)$ is a collection of disjoint
arcs in \((D_1)f\). Thus, there is a disk component of 
\( (D_1)f - (\bigcup_j D_j) \) whose boundary circle is formed exactly by 
one arc \( \alpha \) from \((r_1)f\) and one arc \( \beta \) from \((D_1)f \cap D_s\) for 
some \(s\) (Figure 2.11). In the planar representation of \(H_g\), 
the disk \(D_s\) and the arc \( \beta \) have two copies \(D'_s\), \(D''_s\), and \( \beta'\) 
and \( \beta'' \) for each of them, and one of the arcs \( \beta' \) and \( \beta'' \), 
e.g. \( \beta' \), together with the arc \( \alpha \) forms a simple closed 
curve.

![Figure 2.11](image)

Consider the two boundary circles of an annular 
neighborhood of \(D'_s \cup \alpha\) in the representation plane, there 
is one and only one of them, denoted by \(\gamma\), separating \(D'_s\) 
and \(D''_s\) in two parts. By Lemma 2.5, we may replace \(r_s\) by 
\(\gamma\) without changing other \(r_i\)'s by composing some mapping 
classes generated by \(M, T, \phi, \psi, \theta\) and \(\eta\). Since 
\(#(\gamma \cap (r_1)f) \leq #(r_s \cap (r_1)f) - 2\), and 
\(#(\gamma \cap (r_j)f) \leq #(r_s \cap (r_j)f),\) for \(j \geq 2\), the number \(k\) has been reduced 
by at least two. This completes our lemma.

From Lemmas 2.4, 2.6 and 2.7, we conclude that,
Theorem 2.8. The subgroup $K_g$ is generated by $M$, $T$, $\phi$, $\psi$, $\theta$ and $\eta$.

Proof of Theorem 2.1. All we need is to give the relations between the generators claimed in Theorem 2.1 and the mapping classes $M$, $T$, $\phi$, $\psi$, $\theta$ and $\eta$. By Theorem 1.1 and using some relations from the paper [5], we have the following equations:

$$M = N^2LN,$$
$$p^2 = M \cdot LN^2LN \cdot M,$$
$$\phi = p^2,$$
$$\psi = p^4N^3,$$
$$\theta = TP^2T \cdot (PQN)^{-1} \cdot p^2,$$
$$\eta = TP^2T \cdot (QNP)^{-1},$$
$$PQN = N^3 \cdot PN^2P \cdot P^2 \cdot PNPN,$$
$$QNP = N^3P^2 \cdot (PN^2P)^{-1} \cdot P^2,$$
$$PN^2P = LNLNLN^2LNLLNL = LN^3LN^2LN^2LN^2LN^2LN = LN^3LN^2LN^2LN = (NLN^2)^{-1} \cdot N^6M,$$
and

$$PNPN = M \cdot LN^2LN^2LN^2LN^2 = M \cdot LN^2LN^2LN^2LN^2 \cdot N^6 = M \cdot LN^2LN \cdot N^3 \cdot NLN^2L.$$

By Theorem 2.8 and by the above formulas, Theorem 2.1 is obvious.

Remark. The topological explanation of the generators of $K_g$ is very clear. $M$ is the $360^\circ$-twist along the meridian circle $a_1$, $p^2$ and $N^3$ are the $180^\circ$-twists along the circles $[a_1,b_1]$ and $[a_1,b_1][a_2,b_2]$ respectively, $T$ totates
the handles, and $\overline{N}^3P_2^P = \overline{N}^2L_2^3 \cdot \overline{N}_2L_2$. $M$ is a composition of Dehn twists along the curves $a_1, b_1\overline{a}_1\overline{b}_1a_2\overline{b}_2$ and $b_2$, and is also obtained by sliding one foot of the first handle around the longitude $b_2$ of the second handle (Figure 2.12).

![Figure 2.12](image)

3. Heegaard Splitting of the 3-Sphere $S^3$

Let $F_g$ be the closed orientable surface of genus $g$ embedded unknottedly in $S^3$ and bounding two handlebodies $H_g$ and $H'_g$. Let $\mathcal{B} = \{a_1, b_1, a_2, b_2, \ldots, a_g, b_g\}$ be a system of base curves on the surface $F_g$ based at a basepoint $O$, such that $a_1$'s are meridians of the handlebody $H_g$, and $b_1$'s are meridians of the handlebody $H'_g$. Let $K_g$ and $K'_g$ denote the subgroups of the group $M_g$ formed by the mapping classes which can extend to the solid handlebodies $H_g$ and $H'_g$. 
respectively. For any mapping class \( f \) of \( Mg \), we will denote by

\[
M_f = H^f \cup H^g
\]

the closed 3-manifold associated by \( f \), formed by identifying each point \( X \) of \( \partial H^f \) with the point \( (X)f \) of \( \partial H^g \). It is easy to see,

**Proposition 3.1.** For any mapping classes \( f \in Mg, h \in Kg \) and \( h' \in Kg' \),

\[
M_f = M_{h'h'}. 
\]

In particular, Waldhausen ([8]) proved that,

**Theorem 3.2.** Any genus-\( g \) Heegaard splitting of the 3-sphere is an element of the semiproduct of subgroups, \( Kg' \cdot Kg \).

We obtained a specific description of \( Kg \) in the last section, now we need one for \( Kg' \). In fact, if \( \varphi \) is a homeotopy class induced by a homeomorphism from the handlebody \( H^f \) onto the handlebody \( H^g \), then \( Kg' = Kg/\varphi \). We will call such a homeotopy class a transfer operation. For example,

**Examples 3.3.**

(1) the reversion map \( R \) is a transfer operation, since

\[
(a_i)^R = b_{g-i+2(\text{mod } g)}', \quad \text{and} \quad (b_i)^R = a_{g-i+2(\text{mod } g)}',
\]

for any \( i = 1,2,\ldots,g \).
(2) the homeotopy class \( \pi = (PT)^g_{PT} = P_1 P_2 \ldots P_g \) is a transfer operation, where \( P_i = T_i^{-1} P T_i^{-1} \), since

\[
(a_i)\pi = a_i b_i \bar{a}_i, \text{ and } (b_i)\pi = \bar{a}_i,
\]

for any \( i = 1, 2, \ldots, g \).

Using the homeotopy class \( \pi \), we have,

**Proposition 3.4.** The subgroup \( K'_g = \pi K_g \pi \) is generated by the mapping classes \( T, N^3, P^2, \ \text{PN}^2 P \) and \( L \).

**Proof.** The proposition is an obvious consequence of the following formulas:

\[
\begin{align*}
\pi M & = PMF = LPF = L, \\
\pi T & = P_1 P_2 \ldots P_g T P \ldots \bar{P} \bar{F} = P_1 P_2 \ldots P g \bar{P} 1 g \ldots \bar{P} \bar{F} T = T, \\
\pi N^3 & = P_1 P_2 N^3 \bar{F} \bar{F} = P \dot{P} N^3 \dot{P} \dot{N}^3 \dot{P} N^3 \dot{F} N^3 = N^3, \\
\pi P^2 & = P_1 \cdot P^2 \cdot \bar{F} \bar{F} = P^2,
\end{align*}
\]

and
\[
\begin{align*}
\pi & = P_1 P_2 \cdot P^2 \cdot \bar{P} \bar{F} \bar{F} = P^2 \cdot P_2 N^2 \bar{F} \\
& = P^2 N^3 \cdot \dot{P} N^2 \dot{P} \dot{F} N^3.
\end{align*}
\]

Denote by \( N \) the subgroup of \( M_g \) generated by the elements \( T, P^2, N^3 \) and \( \text{PN}^2 P \), which obviously is a subgroup of \( K'_g \cap K_g \). Using a result of Powell [6] that the subgroup \( K'_g \cap K_g \) is generated by \( T, N^3, P^2, \omega \) and \( \eta \), we have the following consequence:

**Corollary 3.5.**

\( N = K'_g \cap K_g \).
Theorem 3.6. The associated 3-manifold $M_f$ of a mapping class $f$ is the 3-sphere $S^3$ if and only if $f$ is an element of the set
\[(L,N) \cdot (N,M).\]

Before we end this section, we discuss some more relations among the mapping classes in those subgroups.

Let $L_i = T_i^{-1}LT_i^{-1}$, and $M_i = T_i^{-1}MT_i^{-1}$, for $i = 1,2,\ldots,g$. Let $L$ and $M$ denote the abelian subgroups of rank $g$ generated by the $L_i$'s and $M_i$'s respectively.

Proposition 3.7. For any $i = 1,2,\ldots,g$,

\[(a) \quad L_i T = TL_i^{-1}, \quad M_i T = TM_i^{-1},\]
\[(b) \quad L_i P^2 = P^2 L_i, \quad M_i P^2 = P^2 M_i,\]
\[(c) \quad L_i N^3 = N^3 L_i, \quad M_i N^3 = N^3 M_i, \text{ for } i \neq 1,2,\]
\[L_1 N^3 = N^3 L_2, \quad M_1 N^3 = N^3 M_2,\]
\[L_2 N^3 = N^3 L_1, \quad M_2 N^3 = N^3 M_1,\]
\[(d) \quad L_i PN^2 P = PN^2 PL_i, \quad M_i PN^2 P = PN^2 PM_i, \text{ for } i \neq 1,2,\]
and
\[L_1 PN^2 P = PN^2 PL_2, \quad M_2 PN^2 P = PN^2 PM_1.\]

Proof. Since
\[T = [a_g, b_g, a_1, b_1, \ldots, a_{g-1}, b_{g-1}],\]
\[N^3 = [xa_2 \bar{x}, xb_2 \bar{x}, a_1, b_1, a_3, b_3, \ldots, a_g, b_g],\]
\[P^2 = [c_1 \bar{a}_1, c_1 \bar{b}_1, a_2, b_2, \ldots, a_g, b_g],\]
and
\[PN^2 P = [xb_2 \bar{x}, a_2, b_2, a_3, b_3, \ldots, a_g, b_g],\]
the proposition is clear.
Proposition 3.8. \( L \cap N = 1, \) and \( M \cap N = 1. \)

Proof. Consider the image of \( L \) and \( N \) in Siegel's modular group ([2]). For any element \( f \in N \cdot f \) leaves the subspace \( \mathbb{Z}^g \) generated by the \( \alpha_i \)'s in \( H_1(F_g;\mathbb{Z}) \cong \mathbb{Z}^2g \) invariant, by looking at the expressions in the proof of the last proposition. But the only element of \( L \) having this property is the identity. Therefore \( L \cap N = 1. \) And analogously, \( M \cap N = 1. \)

References


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