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COUNTABLE COVERS OF SPACES BY MIGRANT SETS

Zoltan Balogh, Joe Masburn, and Peter Nyikos

The motivation for this note is a paper by Hidenori Tanaka [T] in which he shows that the Pixley-Roy hyperspace of a metric space X is normal if and only if X is an almost strong q-set. He defined an almost strong q-set to be a space X such that for every $n \in \omega$ with $n \neq 0$, every symmetric subset of X^n is an F_σ subset of X^n. Here symmetric means that the set is closed with respect to permutations of coordinates. Earlier, Przymusinski and Tall [PT] and Rudin [R] had combined to show that the Pixley-Roy hyperspace of a separable metric space X is normal if and only if X is a strong q-set. The obvious question is "When is an almost strong q-set going to be a strong q-set?" Tanaka's partial answer was that this will happen in strongly zero-dimensional metric spaces. He actually showed that a linearly ordered space is a strong q-set if and only if it is an almost strong q-set. He did this by finding an open subset U of X, namely, $\{\langle x_1, x_2, \ldots, x_n\rangle : x_1 < x_2 < \ldots < x_n\}$, with the property that if π is any permutation of coordinates then $U \cap \pi(U) = \emptyset$. But his proof raises some other questions. For let $Y = \{\langle x_1, x_2, \ldots, x_n\rangle : \forall i, j \in \{1, \ldots, n\}(i \neq j \Rightarrow x_i \neq x_j)\}$, and let Π be the group of permutations of coordinates of elements of Y. Then $\{\pi(U): \pi \in \Pi\}$ is a finite open cover of Y with the property that if V is an element of this cover and π is a nonidentity element of Π then $V \cap \pi(V) = \emptyset$. Is it possible to find such covers of metric spaces for other finite groups? If the answer to this question is yes, then the elements of the group can be homeomorphisms rather than isometries, since the space can be remetrized by $d'(x,y) = \max\{d(g(x),g(y)): g \in G\}$ and this new metric makes the elements of G isometries. Must the groups be finite? Can spaces other than metric spaces be used? In this note we will answer some of these questions.

Let G be a group of bijections from a set X to itself. Throughout this paper, e will denote the identity of the group under consideration. A subset Y of X is G-*migrant* if $Y \cap g(Y) = \emptyset$ for all $g \in G\setminus\{e\}$. Note that Y is G-migrant if and only if for any $g,h \in G$, if $g \neq h$ then $g(Y) \cap h(Y) = \emptyset$. If it is clear which group is being considered, we will call such a set migrant. Using the terminology of Conner and Floyd [CF], we will say that G acts freely on X if every element of $G\setminus\{e\}$ is fixed point free.

Lemma 1. *Let* X *be a normal space and let* G *be a finite group of autohomeomorphisms of* X. *If* F *is a closed migrant subset of* X *then there is an open migrant subset* U *of* F *such that* $F \subseteq U$.

Proof. For each $g \in G$ let U_g be an open neighborhood of g(F) such that $U_g \cap U_h \neq \emptyset$ if an only if $g = h$. Let

$U = \bigcap_{g \in G} g^{-1} (U_g)$. Then $F \subseteq U$, and $g(U) \subseteq U_g$ for all $g \in G$.

This proof can be easily modified to show that every element of a Hausdorff space has a migrant neighborhood, which gives us the following proposition.

Proposition 2. *If* X *is a* T_2 *Lindelöf space and* G *is a finite group of autohomeomorphisms acting freely on* X, *then* X *has a countable open migrant cover.*

The general question of what sort of space will have countable open or closed migrant covers is not as easy as Proposition 2, as we can see by this example, which is a modification of Heath's tangent V space.

Example 1. Let $X = \{\langle p,q\rangle : p,q \in \mathbb{R},\ q \geq 0$, and $p \neq 0\}$, i.e., the upper-half plane without the y-axis. Declare all points of the form $\langle p,q\rangle$ with $q > 0$ to be isolated. For each $p \in \mathbb{R}$ with $p > 0$ and each $n \in \omega$, let $B(p,n)$ be the set of points in X which lie on the line $y = x - p$ and are within 2^{-n} of $\langle p,0\rangle$ together with the points in X other than $\langle -p,0\rangle$ which lie on the line $y = x + p$ and are within 2^{-n} of $\langle -p,0\rangle$. For each $p \in \mathbb{R}$ with $p < 0$ and each $n \in \omega$ let $B(p,n)$ be the set of points in X which lie on the line $y = p - x$ and are within 2^{-n} of $\langle p,0\rangle$ together with the set of points in X other than $\langle -p,0\rangle$ which lie on the line $y = -p - x$ and are within 2^{-n} of $\langle -p,0\rangle$. Give X the topology generated by these neighborhood bases. Then X has a uniform basis and is therefore a metacompact Moore space. Define $g: X \to X$ by $g(p,q) = \langle -p,q\rangle$. Clearly g is a homeomorphism and has no fixed points. Let $G = \{e,g\}$.

Let U be an open migrant subset of X and let A be the intersection of U with the x-axis. We will show that A is σ-discrete in the usual topology on $\mathbb{R}$, and is therefore countable. For each $\langle p,0\rangle \in A$ let $n_p \in \omega$ such that $B(p,n_p) \subseteq U$. Assume that there is $n \in \omega$ such that $D_n = \{\langle p,0\rangle \in A: n_p \leq n\}$ is not discrete in the usual topology on $\mathbb{R}$, and let $\langle p,0\rangle \in D_n$ be a limit point of D_n. We may assume that $p > 0$. Since $\langle p,0\rangle$ is a limit point of D_n there is a monotone sequence of elements of D_n which converges to $\langle p,0\rangle$. If the sequence is decreasing, we may choose an element $\langle q,0\rangle$ of D_n such that the lines $y = x + p$ and $y = -x - q$ intersect at a point $\langle a,b\rangle$ which is within 2^{-n_p} of $\langle -p,0\rangle$ and within 2^{-n_q} of $\langle -q,0\rangle$. Then $\langle a,b\rangle \in B(p,n_p)$ and $\langle -a,b\rangle \in B(q,n_q)$, contradicting the assumption that U is migrant. If the sequence is increasing, we may choose elements $\langle q,0\rangle$ and $\langle r,0\rangle$ of D_n such that the lines $y = x + q$ and $y = -x - r$ intersect at a point $\langle a,b\rangle$ which is within 2^{-n_q} of $\langle -q,0\rangle$ and within 2^{-n_r} of $\langle -r,0\rangle$. Then $\langle a,b\rangle \in B(q,n_q)$ and $\langle -a,b\rangle \in B(r,n_r)$, contradicting the assumption that U is migrant. It follows that X cannot be covered by less than $\mathfrak{c}$ migrant open sets.

Let $\mathcal{F}$ be a finite collection of subsets of X such that if $p,q \in \mathbb{R}$ with $p,q > 0$ then $\langle p,q\rangle \in \cup\mathcal{F}$. Then every point in X that lies on the x-axis is in the closure of some element of $\mathcal{F}$. For every $p \in \mathbf{R}$ with $p \neq 0$ let $\mathcal{F}_p = \{F \in \mathcal{F}: \langle p,0\rangle \in \overline{F}\}$. We claim that there is $p \in \mathbf{R}$ such that $\mathcal{F}_p \cap \mathcal{F}_{-p} \neq \emptyset$. Assume not. For each $p \in \mathbf{R}$ let n_p be the least $m \in \omega$ such that $B(p,m)$ meets only elements of $\mathcal{F}_p$ and $B(-p,m)$ meets only elements of $\mathcal{F}_{-p}$. For every $\mathcal{G} \subseteq \mathcal{F}$ and every $n \in \omega$ let $D(\mathcal{G},n) = \{p \in \mathbf{R}: p > 0, \mathcal{F}_p = \mathcal{G}, n_p \leq n,$ and $n_{-p} \leq n\}$. Assume that $\langle p,0\rangle$ is a limit point of $D(\mathcal{G},n)$ in the usual topology on $\mathbf{R}$. Again, there is a monotone sequence in $D(\mathcal{G},n)$ which converges to $\langle p,0\rangle$. If the sequence is increasing, we may choose an element q of $D(\mathcal{G},n)$ such that the lines $y = p - x$ and $y = x - q$ intersect at a point within 2^{-n} of $\langle p,0\rangle$ and within 2^{-n} of $\langle q,0\rangle$. This point of intersection cannot be in an element of either $\mathcal{F}_q$ or $\mathcal{F}_{-q}$ and therefore is not in any element of $\mathcal{F}$. If the sequence is decreasing, we may choose an element q of $D(\mathcal{G},n)$ such that the lines $y = x - p$ and $y = q - x$ intersect at a point within 2^{-n} of $\langle p,0\rangle$ and within 2^{-n} of $\langle q,0\rangle$. This point of intersection cannot be in any element of $\mathcal{F}_p$ or of $\mathcal{F}_{-q}$ and so is not in any element of $\mathcal{F}$. It follows that each $D(\mathcal{G},n)$ is countable, a contradiction. Thus there is $p \in \mathbf{R}$ such that $\mathcal{F}_p \cap \mathcal{F}_{-p} \neq \emptyset$. Let $F \in \mathcal{F}_p \cap \mathcal{F}_{-p}$. Then $\langle p,0\rangle \in \overline{F}$ and $\langle -p,0\rangle \in \overline{F}$ and $\overline{F}$ is not migrant. Therefore X cannot be covered by a finite number of closed migrant sets.

But X can be covered by a countable number of closed migrant sets, which means by Lemma 1 that X cannot be normal. Indeed, we shall see that this type of space cannot be normal.

Theorem 3. *If* X *is a paracompact space and* G *is a finite group of autohomeomorphisms acting freely on* X, *then* X *has a countable closed (or open) migrant cover.*

Proof. We will induct on $|G|$. If $|G| = 1$, then X is migrant. Let us assume that $|G| = n$ for some $n > 1$, and that for every paracompact space and every finite group of cardinality less than n the conclusion of the theorem holds. We will prove two lemmas.

Lemma 4. *Let* U *be an open subset of* X *and let* $\{B_\gamma : \gamma \in \Gamma\}$ *be a discrete collection of open migrant* F_σ *subsets of X. If* $U = \cup_{\gamma\in\Gamma} B_\gamma$ *and* $g(U) = U$ *for every* $g \in G$, *then there is an open migrant* F_σ *subset* V *of* X *such that* $\cup_{g\in G} g(V) = U$.

Proof. Set $G = \{g_m : m \leq n\}$. For every $\vec{\gamma} = \langle \gamma_1, \ldots, \gamma_n \rangle \in \Gamma^n$, let $C(\vec{\gamma}) = \cap_{i=1}^n g_i(B_{\gamma_i})$. The sets $C(\vec{\gamma})$ ($\vec{\gamma} \in \Gamma^n$) have the following properties:

(1) if $C(\vec{\gamma}) \cap C(\vec{\beta}) \neq \emptyset$, then $\vec{\gamma} = \vec{\beta}$;

(2) $\cup_{\vec{\gamma}\in\Gamma^n} C(\vec{\gamma}) = U$;

(3) if $g \in G\setminus\{e\}$ and $\vec{\gamma} \in \Gamma^n$, then there is a $\vec{\beta} \in \Gamma^n\setminus\{\vec{\gamma}\}$, such that $g(C(\vec{\gamma})) = C(\vec{\beta})$.

To see that property (1) is true, note that if $\gamma_i \neq \beta_i$, then $C(\vec{\gamma}) \cap C(\vec{\beta}) \subseteq g_i(B_{\gamma i}) \cap g_i(B_{\beta i}) = \emptyset$. Property (2) follows from the fact that $g\restriction U$ is a bijection for every $g \in G$. To verify property (3), note that $g(C(\vec{\gamma})) = \cap_{i=1}^n (g \circ g_i)(B_{\gamma_i})$. Since $G = \{g \circ g_i : 1 \leq i \leq n\}$, there is a permutation π of $\{1,\ldots,n\}$ such that $g \circ g_i = g_{\pi(i)}$ for all $i \in \{1,\ldots,n\}$. So $g(C(\vec{\gamma})) = \cap_{i=1}^{n} g_{\pi(i)}(B_{\gamma_i}) = C(\vec{\beta})$, where $\beta_i = \gamma_{\pi^{-1}(i)}$.

Observing further that each $C(\vec{\gamma})$ is an open F_σ subset of X, properties (1), (2), and (3) imply that $\mathcal{P} = \{C(\vec{\gamma})\colon \vec{\gamma} \in \Gamma^n \text{ and } C(\vec{\gamma}) \neq \emptyset\}$ is a discrete cover of U by open F_σ sets such that for every $g \in G\setminus\{e\}$ and every $C(\vec{\gamma}) \in \mathcal{P}$, $g(C(\vec{\gamma})) \in \mathcal{P}\setminus\{C(\vec{\gamma})\}$. This implies the conclusion of Lemma 4.

Lemma 5. *Let* $\mathcal{F}$ *be a countable family of* F_σ *subsets of* X *such that for every* $F \in \mathcal{F}$,

(1) *if* $g \in G$ *and* $F \cap g(F) \neq \emptyset$, *then* $g(F) = F$;

(2) *there is a* $g \in G$ *such that* $g(F) \neq F$.

Then $\cup\mathcal{F}$ *is the union of a countable family of closed migrant subsets of* X.

Proof. Let $Y = \cup\mathcal{F}$. Note that for every $F \in \mathcal{F}$, $G(F) = \{g \in G\colon g(F) = F\}$ is a subgroup of G with a smaller order than G. By our inductive hypothesis, there is a countable family $\mathcal{H}_F$ of closed $G(F)$-migrant subsets of the subspace F which covers F. Observe that by property (1), for $H \in \mathcal{H}_F$ and $g \notin G(F)$, $H \cap g(H) = \emptyset$. Therefore, since closed subsets of F_σ subsets of X are F_σ subsets of X, $\mathcal{H} = \cup_{F\in\mathcal{F}}\mathcal{H}_F$ is a countable family of F_σ subsets of X, each of which is migrant. This, of course, gives rise to a countable cover of Y consisting of closed migrant subsets of X.

We now return to the proof of Theorem 3. By Lemma 1, X has an open cover consisting of migrant sets. Let $\mathcal{B} = \cup_{m\in\omega}\mathcal{B}_m$ be a σ-discrete open refinement of this cover by F_σ-sets. Clearly every element of $\mathcal{B}$ is migrant. For every $m \in \omega$, let $U_m = \cup\mathcal{B}_m$. Note that each U_m is an F_σ-set. For every $m \in \omega$ and every $A \subseteq G$, let $F_m(A) = \cap_{g\in A} g(U_m)\setminus\cup_{g\notin A} g(U_m)$ and set $\mathcal{F}_m = \{F_m(A)\colon A \neq G\}$. Then $g(F_m(A)) = F_m(gA)$ for all $g \in G$ and all $A \subseteq G$. Thus $F_m(A) \cap F_m(gA) \neq \emptyset$ implies $A = gA$, which implies that $F_m(A) = F_m(gA)$. So $\mathcal{F}_m$ satisfies the conditions of Lemma 5 and $\cup\mathcal{F}_m$ is the union of countably many closed subsets of X.

Finally, let us consider $F_m(G) = \cap_{g\in G} g(U_m)$. We will show that this set satisfies the conditions of Lemma 4. As in the proof of Lemma 4, let $G = \{g_i\colon i = 1,\dots,n\}$. Let $\mathcal{B}_m = \{B_\delta\colon \delta \in \Delta_m\}$. For each $\vec{\delta} \in \Delta_m^n$, let $V_{\vec{\delta}} = \cap_{i=1}^n g_i(B_{\delta_i})$. Then $\{V_{\vec{\delta}}\colon \vec{\delta} \in \Delta_m^n\}$ is a discrete collection of open migrant F_σ subsets of X and $F_m(G) = \cup_{\vec{\delta}\in\Delta_m^n} V_{\vec{\delta}}$. Also, $g(F_m(G)) = F_m(G)$ for all $g \in G$. Thus $F_m(G) \cup (\cup\mathcal{F}_m) \supseteq U_m$ is the union of countably many closed migrant subsets of X. It follows that X is also. The fact that X can be covered by a countable number of open migrant sets follows from Lemma 1.

Question. If G is a finite group of autohomeomorphisms acting freely on a subparacompact space X, will X necessarily have a countable closed migrant cover? What if X is a Moore space or a σ-space?

In order to show that an almost strong q-set, X, is a strong q-set we will need to make use of migrant subsets of X^n for any n. But then we need to know that each finite power of X is paracompact. For that reason we will consider paracompact Σ-spaces. Then every finite power is again a paracompact Σ-space. Note that if X is an almost strong q-set then every symmetric subset of X^2 is an G_δ subset of X^2. In particular, an almost strong q-set has a G_δ diagonal. Thus every Σ-space which is an almost strong q-set is a σ-space.

Theorem 6. *Let* X *be a paracompact* Σ*-space. If* X *is an almost strong* q*-set, then it is a strong* q*-set.*

Proof. The proof proceeds by induction on the power of X. Let X be an almost strong q-set. Then every subset of X is an F_σ subset of X. Let $n \in \omega$ with $n \geq 1$ and assume that if $m \leq n$ and A is a subset of X^m, then A is an F_σ subset of X^m. Set Y equal to the set of all $\langle x_1, \ldots, x_{n+1}\rangle \in X^{n+1}$ such that for all $i,j \in \{1,\ldots,n+1\}$, $i \neq j$ if and only if $x_i \neq x_j$. Then $X^{n+1}\setminus Y$ is the union of a finite number of closed subsets of X^{n+1}, each of which is homeomorphic to X^m for some $m \leq n$. Thus, we need only show that every subset of Y is an F_σ subset of X^{n+1}. Since X^{n+1} is a paracompact σ-space, Y is paracompact. Let P be the group of permutations on $\{1,\ldots,n+1\}$. For each $\pi \in P$ and each $\vec{x} = \langle x_1,\ldots,x_{n+1}\rangle \in Y$ let $g_\pi(\vec{x}) = \langle x_{\pi(1)}, \ldots, x_{\pi(n+1)}\rangle$. Then $G = \{g_\pi\colon \pi \in P\}$ is a group of autohomeomorphisms acting freely on Y. By Theorem 4, there is a countable open cover $\mathcal{U}$ of Y consisting of migrant sets. Now we need only show that if $U \in \mathcal{U}$, then every subset of U is an F_σ-subset of X^{n+1}. Let $U \in \mathcal{U}$ and $A \subseteq U$. Then $B = \cup_{g\in G} g(A)$ is a symmetric subset of X^{n+1} and is therefore an F_σ-subset of X^{n+1}. It follows that $A = B\setminus \cup\{g(U): g \in G\setminus\{e\}\}$ is an F_σ-subset of X^{n+1}.

In view of Tanaka's theorem [T] that for a metrizable space X, the normality or hereditary countable paracompactness of PR[X] is equivalent to X being an almost strong q-set, Theorem 5 has the following corollary. Here PR[X] denotes the Pixley-Roy hyperspace of X.

Corollary 7. *Let* X *be a metric space. The following are equivalent.*

(1) PR[X] *is normal.*

(2) PR[X] *is hereditarily countably paracompact.*

(3) X *is a strong* q-*set.*

Is it possible in Theorem 3 to make G infinite? The following example shows that indeed it cannot. Let X be a circle with the usual topology. Let G be the group of rotations of X by angles which are rational multiples of π. Then G is a countable group of auto-isomitries acting freely on X. Also, $\{g^n(x): n \in \omega\}$ is finite for every $g \in G$ and every $x \in X$. But every open subset of X has two points which differ by an angle which is a rational multiple of π, so every migrant set must have empty interior.

Thus the conclusion of Lemma 1 does not hold, and X cannot be covered by a countable number of closed migrant sets.

Let us now consider the question of how many migrant sets are needed to cover a space X. The only reason that the collection $\{B_\gamma: \gamma \in \Gamma\}$ in Lemma 4 had to be locally finite was to make the set U an F_σ-set. We can delete this requirement and still have the following lemma. If $\{B_\gamma: \gamma \in \Gamma\}$ is a pairwise disjoint collection of migrant open subsets of a space X, G is a finite group of autohomeomorphisms acting freely on X, $U = \cup_{\gamma\in\Gamma} B_\gamma$, and $g(U) = U$ for all $g \in G$, then there is a migrant open set V of X such that $U = \cup_{g\in G} g(V)$. In particular, if U = X then V is clopen. This variation of Lemma 4 shows that if G is a finite group of autohomeomorphisms acting freely on an ultraparacompact space X, then there is a clopen subset U of X such that $X = \cup_{g\in G} g(U)$. In a slightly more general setting we have the following proposition.

Proposition 8. *If* G *is a finite group of autohomeomorphisms acting freely on an ultranormal space* X *and* X *has a countable migrant closed cover, then there is a clopen subset* U *of* X *such that* $X = \cup_{g\in G} g(U)$.

Proof. Let $C = \{C_n: n \in \omega\}$ be a migrant closed cover of X. For each $n \in \omega$ let U_n be a clopen migrant set such that $C_n \subseteq U_n$. Set $V_0 = U_0$ and, for every $n \in \omega$, let $V_{n+1} = U_{n+1} \setminus \cup_{i\in n+1} U_i$. Then $\{V_n: n \in \omega\}$ is a pairwise disjoint migrant clopen cover of X.

Proposition 9. *Let* G *be a finite group of autohomeomorphisms acting freely on an expandable ultranormal space* X. *If* X *has a* σ*-locally finite closed migrant cover, then there is a clopen subset* U *of* X *such that* $X = \cup_{g\in G} g(U)$.

Proof. Let $\mathcal{C}$ be a σ-locally finite closed migrant cover of X, and expand $\mathcal{C}$ to a σ-locally finite clopen migrant cover $\mathcal{U} = \cup_{n\in\omega}\mathcal{U}_n$. For $n \in \omega$ let $\mathcal{U}_n = \{U_{n\alpha}: \alpha \in \kappa_n\}$, where κ_n is some indexing cardinal. For $\beta \in \kappa_n$ let $V_{n\beta} = U_{n\beta}\setminus\cup_{\alpha<\beta}U_{n\alpha}$. Then $\mathcal{V}_n = \{V_{n\alpha}: \alpha \in \kappa_n\}$ is a locally finite, pairwise disjoint clopen collection of migrant sets, and $\mathcal{V} = \cup_{n\in\omega}\mathcal{V}_n$ covers X. Let $\mathcal{W}_0 = \mathcal{V}_0$ and for each $n \in \omega$ let $\mathcal{W}_{n+1} = \{V_{n+1,\alpha}\setminus\cup_{j\in n+1}(\cup\mathcal{W}_j): \alpha \in \kappa_{n+1}\}$. Then $\mathcal{W} = \cup_{n\in\omega}\mathcal{W}_n$ is a pairwise disjoint clopen migrant cover of X.

Another class of spaces which have finite clopen migrant covers is the class of extremally disconnected spaces. Every space has a maximal migrant open subset. This set along with its translates will be dense in the space. If the space is also extremally disconnected, then the closures of these sets will form a pairwise disjoint clopen migrant cover of size $|G|$.

We have already seen in Example 1 that not all spaces can be covered by a finite or even countable number of migrant open sets. The next example shows that even a metric space need not have a finite open migrant cover.

Example 2. Let X be the disjoint sum of all n-spheres for $n \in \omega$. Let g be the mapping of X to itself which takes each point of S^n to its antipodal point. Then X is paracompact and {e,g} is a group of autohomeomorphisms acting freely on X, but X cannot be covered by a finite number of migrant open (or closed) sets by the Borsuk-Ulam Theorem, as Bill Weiss and K. P. Hart pointed out. This space is infinite dimensional while the spaces of Propositions 8 and 9 are zero dimensional. So it seems reasonable that the dimension of a space might have something to do with the size of its migrant open covers, as, in fact, it does.

Theorem 10. *Let* X *be a paracompact space with* Ind $X \leq n$, *and in which the closed subspaces satisfy the countable closed sum theorem for* Ind. *If* G *is a finite group of autohomeomorphisms acting freely on* X *then there is a collection* $\mathcal{M}$ *of migrant open (or closed) subsets of* X *such that* $|\mathcal{M}| \leq n + 1$ *and* $X = \cup_{M \in \mathcal{M}} \cup_{g \in G} g(M)$.

Proof. If $n = 0$ then X is ultraparacompact. Assume that the conclusion of the theorem holds for any space Y with Ind $Y \leq n$, and that Ind $X = n + 1$. Using paracompactness and dimension, we may obtain a locally finite, σ-discrete open migrant cover $\mathcal{U} = \cup_{m \in \omega} \mathcal{U}_m$ of X such that Ind Bd$U \leq n$ for all $U \in \mathcal{U}$.

For each $m \in \omega$ let $B_m = \cup\{\text{Bd}U\colon U \in \mathcal{U}_m\}$. Then B_m is closed in X and Ind $B_m \leq n$. Let $\mathcal{V}_m = \{g(U)\colon U \in \mathcal{U}_m$ and $g \in G\}$ and let $V_m = \cup\mathcal{V}_m$. Let $C_m = (\cup_{g \in G} g(B_m)) \setminus \cup\{V_k\colon k < m\}$. Now $\cup_{g \in G} g(B_m)$ is closed in X so Ind $\cup_{g \in G} g(B_m) \leq n$. The set C_m is closed in $\cup_{g \in G} g(B_m)$ so Ind $C_m \leq n$. Also,

$\overline{V}_m \subseteq V_m \cup C_m \cup(\cup_{k\in m}V_k)$ and $\{C_m: m \in \omega\}$ is locally finite. Thus $C = \cup_{m\in\omega}C_m$ is closed and Ind $C \leq n$. Also, $g(C) = C$ for all $g \in G$, so there is a family of no more than n migrant closed subsets of C whose translates under G cover C. Call this collection $N = \{N_j: j \in n\}$. Expand N to a collection of n migrant open sets $\mathcal{O}$.

For every $m \in \omega$ let $W_m = (V_m \setminus C_m) \setminus \cup_{k\in m}\overline{V}_k$. Then $\{W_m: m \in \omega\}$ is a pairwise disjoint collection of open subsets of X such that $g(W_m) = W_m$ for all $m \in \omega$ and $X\setminus C = \cup_{m\in\omega}W_m$. We will show that for every $m \in \omega$ there is a migrant open subset W_m' of X such that $W_m = \cup_{g\in G}g(W_m')$. For each $p \in W_m$ let $\mathcal{V}_m(p) = \{V \in \mathcal{V}_m: p \in V\}$. For every $g \in G$ there is at most one $U \in \mathcal{U}_m$ such that $p \in g(U)$, so $|\mathcal{V}_m(p)| \leq |G|$. Also, $\mathcal{V}_m(p) \neq \emptyset$ since $p \in W_m$. Let $W_m(p) = \cap\mathcal{V}_m(p)\setminus\cup\{\overline{V}: V \in \mathcal{V}_m\setminus\mathcal{V}_m(p)\}$. Then $W_m(p)$ is an open migrant subset of X. If $p,q \in W_m$ and $W_m(p) \cap W_m(q) \neq \emptyset$, then $\mathcal{V}_m(p) = \mathcal{V}_m(q)$ so $W_m(p) = W_m(q)$. Thus $\{W_m(p): p \in W_m\}$ partitions W_m into open sets which are permuted by G. Now pick one element from the orbit of every member of this set and take their union to form W_m'. Set $W = \cup_{m\in\omega}W_m'$. Then $\mathcal{M} = \mathcal{O} \cup \{W\}$ is a collection of open migrant subsets of X such that $|\mathcal{M}| \leq n + 1$ and $X = \cup_{M\in\mathcal{M}}\cup_{g\in G}g(M)$. This can be shrunk to a collection of closed migrant sets having the same properties.

Since dim is the same as Ind in metric spaces and the countable closed sum theorem holds, we have that if G is a finite group of autohomeomorphisms acting freely on a metric space X, then X has an open (or closed) migrant cover of no more than $|G|(\dim X + 1)$ sets. Also, by slightly modifying the proof of Theorem 10, one can make the members of the migrant cover of X be regular closed sets, zero-sets, or cozero-sets.

The number $|G|(\dim X + 1)$ is not optimal, but we do not yet know what the optimal number is. We conjecture that it is $|G| + \dim X$, which is true if $\dim X = 0$ by Proposition 8 or if $|G| = 2$ by [CF].

We will mention one more class of spaces which have finite covers by open or closed migrant sets: normal spaces of finite scattered height. This means that a different approach from that taken in our first example must be used if a normal space without a finite open or closed migrant cover is desired.

Proposition 11. *If* G *is a finite group of autohomeomorphisms acting freely on a normal space* X *of scattered height* n, *for some* $n \in \omega$, *then there is a collection* $\mathcal{M}$ *of open (or closed) migrant subsets of* X *such that* $|\mathcal{M}| \leq n + 1$ *and* $X = \cup_{M\in\mathcal{M}}\cup_{g\in G} g(M)$.

Thus X has an open (or closed) migrant cover of size no greater than $|G|(n + 1)$.

Proof. This is clearly true if X is discrete. Let $n \in \omega$ and assume that if a space has scattered height n, then there is a collection $\mathcal{M}$ of open migrant subsets of X such that $|\mathcal{M}| \leq n + 1$ and $X = \cup_{M\in\mathcal{M}}\cup_{g\in G} g(M)$. Let X =

$\bigcup_{k=0}^{n+1} X_k$ where X_0 is the set of isolated points of X, X_{k+1} is the set of isolated points in $X\setminus\bigcup_{j=0}^{k} X_j$ for $k = 0,\ldots,n$, and X_{n+1} is discrete. Then there is a migrant closed subset C of X_{n+1} such that $X_{n+1} = \bigcup_{g\in G} g(C)$. By normality, expand C to a migrant open subset U of X. Then $X\setminus\bigcup_{g\in G} g(U)$ is a normal space of scattered height n and $\{g\restriction(X\setminus\bigcup_{h\in G} h(U)) : g \in G\}$ is a group of autohomeomorphisms acting freely on $X\setminus\bigcup_{g\in G} g(U)$. By assumption, there is a collection $\mathcal{N}$ of open migrant subsets of $X\setminus\bigcup_{g\in G} g(U)$ such that $|\mathcal{N}| \leq n + 1$ and $X\setminus\bigcup_{g\in G} g(U) = \bigcup_{N\in\mathcal{N}}\bigcup_{g\in g} g(N)$. Then $\mathcal{M} = \mathcal{N} \cup \{U\}$ is the desired collection of open migrant subsets of X. To obtain a collection of closed migrant sets, just shrink the elements of $\mathcal{M}$.

It follows that if X is a normal countably paracompact space of countable scattered height, then it has a countable open and a countable closed migrant cover.

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