$K$-SEMIMETRICS AND 1-CONTINUOUS SEMIMETRICS

by

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A distance function for X is any nonnegative, real-valued function d: X × X → ℝ such that d(x, y) = d(y, x) and d(x, y) = 0 iff x = y for any x, y ∈ X. We use the notation d(x, A] = inf{d(x, y) | y ∈ A}, d[B, A] = inf{d(x, A] | x ∈ B} and Sd(p, ε) = {x ∈ X | d(p, x) < ε}. A distance function d is continuous iff, when d(xn, p) → 0 and d(yn, q) → 0, then d(xn, yn) → d(p, q); it is 1-continuous iff, for any q, when d(xn, p) → 0, then d(xn, q) → d(p, q); and it is developable iff, when d(xn, p) → 0 and d(yn, p) → 0, then d(xn, yn) → 0 (or, equivalently, if d(xn, p) → 0, then (xn) is d-Cauchy).

Any distance function d determines a topology Td = {A ⊆ X | if p ∈ A, then Sd(p, ε) ⊆ A for some ε}, which is called the symmetric topology for X. Thus, d is a symmetric for (X, T) iff T = Td. If, for each p ∈ X, the set of spheres Sd(p, ε) is a neighborhood base for p in (X, T), then we follow convention in saying that d is a semimetric (or an admissible semimetric) for (X, T); a topological space (X, T) is semimetrizable iff there is a semimetric for (X, T). Clearly, if d is a semimetric for (X, T), then T = Td; that the converse need not hold is a well known result of Arhangel'skii (see [4]).

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Finally, when $d$ is a distance function for $X$ such that $T_d \subseteq T$ and $d[A,B] > 0$, when $A$ and $B$ are nonempty, disjoint compact subsets of $(X,T)$ (i.e., $d$ separates disjoint compact subsets of $(X,T)$), then we say that $d$ is a $K$-distance function on $(X,T)$. Similarly, we have the notion of $K$-semimetric, $K$-developable semimetric, etc.

1. Developable semimetrics and $K$-semimetrics

For semimetrizable spaces our study seeks to establish the strongest possible admissible semimetric for a space $(X,T)$.

First, we consider spaces which admit developable semimetrics and $K$-semimetrics. We note that Burke's Example [2; Example 1, p. 126], which we denote as $B_2$, is developable semimetrizable, but no admissible semimetric is a $K$-semimetric. Borges' Example [1; Example 2.4, p. 194], which we denote as $B_{\perp}$, is $K$-semimetrizable but no admissible semimetric is developable; see Remark 2.5.

Recall that, for any infinite, maximal family $R$ of infinite almost disjoint subsets of the set $\mathbb{N}$ of natural numbers, the Isbell-Mrówka space $\Psi_R$ is the set $\mathbb{N} \cup R$ with the topology which, for each $A \in R$, has the sets $U_k(A) = \{A\} \cup \{n \in A \mid k \leq n\}$, $k \in \mathbb{N}$, as a local base, and for each $n \in \mathbb{N}$, has $\{n\}$ as a local base. See [5:5I] for further details. We establish that the spaces $\Psi_R$ admit developable semimetrics and $K$-semimetrics, but none that are simultaneously developable and $K$-semimetrics. (Note that
we have shown in [3] that an analogous result holds in
the case of developable semimetrics and Cauchy complete
semimetrics for $\mathcal{R}$. Namely, there is a developable semi-
metric for $\mathcal{R}$ and there is a Cauchy complete semimetric
for $\mathcal{R}$; however, if $d$ is a developable semimetric for $\mathcal{R}$,
then $d$ is not Cauchy complete.)

**Theorem 1.1.** There is a developable semimetric for
$\mathcal{R}$ and there is a K-semimetric for $\mathcal{R}$; however, if $d$ is a
developable semimetric for $\mathcal{R}$, then $d$ is not a K-semimetric.

**Proof.** The distance function for $\mathbb{N} \cup \mathcal{R}$ with $d(x,y) =
d(y,x) = 2^{-x}$, when $x \in y \in \mathcal{R}$, and, otherwise, $d(x,y) = 1$,
when $x \neq y$, is a K-semimetric for $\mathcal{R}$. Note that $d$ is not
developable since each $A \in \mathcal{R}$, viewed as an increasing se-
quence in $\mathbb{N} \subseteq \mathbb{N} \cup \mathcal{R}$, converges to $A \in \mathcal{R}$, but is not
Cauchy. On the other hand, if we modify this distance
function so that $d(x,y) = |2^{-x} - 2^{-y}|$ for $x,y \in \mathbb{N}$, then
we have a developable semimetric for $\mathcal{R}$.

Finally, we show that a developable semimetric for
$\mathcal{R}$ cannot be a K-semimetric. Suppose that $d$ is any de-
velopable semimetric for $\mathcal{R}$. For any positive $\varepsilon$, there are
at most finitely many $A \in \mathcal{R}$ such that $d[A, \mathbb{N}\setminus A] \geq \varepsilon$.
(Otherwise, choose a sequence $\langle A_n \rangle$ of distinct members of
$\mathbb{N}$ such that $d[A_n, \mathbb{N}\setminus A_n] \geq \varepsilon$. Now, for each $i$, choose
$a_i \in A_i \setminus \bigcup \{A_j | j < i\}$. Note that, for $i \neq j$, $d(a_i, a_j) \geq \varepsilon$. The maximality of $\mathbb{N}$ implies that the sequence $\langle a_n \rangle$ in
$\mathbb{N}$ has a subsequence $\langle b_n \rangle$ which converges to some $B \in \mathbb{R}$.
Thus, \( \langle b_n \rangle \) is a convergent sequence which is not \( d \)-Cauchy. This contradicts that \( d \) is developable.) Now, since \( R \) is uncountable, choose \( A \in R \) such that \( d[A, N \setminus A] = 0 \). There is an increasing sequence \( \langle x_n \rangle \) in \( N \setminus A \) such that \( d(x_n, A] \to 0 \). Again, from the maximality of \( R \), we obtain a subsequence \( \langle b_n \rangle \) of \( \langle x_n \rangle \) which converges to some \( B \in R \). It follows that \( A \cup \{ A \} \) and \( (B \setminus A) \cup \{ B \} \) are disjoint compact sets in \( \Psi_R \) which are not separated by \( d \).

**Remark 1.2.** The critical factor in establishing our result is the failure of \( \Psi_R \) to have a regular \( G_\delta \)-diagonal. This becomes apparent in our next theorem. We show that \( (X, T) \) admits a K-developable semimetric iff \( (X, T) \) is a \( w\Delta \)-space with a regular \( G_\delta \)-diagonal. On the other hand, McArthur [7] has shown that any pseudocompact, completely regular, Hausdorff space \( (X, T) \) with a regular \( G_\delta \)-diagonal is metrizable. It follows that \( \Psi_R \) does not have a regular \( G_\delta \)-diagonal and, therefore, can not admit a K-developable semimetric.

Recall that \( X \) has a \( G_\delta \)-diagonal iff the diagonal of \( X \), \( \Delta_X = \{(x, x) \mid x \in X\} \), is a \( G_\delta \)-set in the product; \( X \) has a regular \( G_\delta \)-diagonal [8] iff \( \Delta_X \) is a countable intersection of regular closed neighborhoods. \( (X, T) \) is a \( w\Delta \)-space iff there is a sequence \( \langle G_n \rangle \) of open covers of \( X \) such that, if \( \langle x_n \rangle \) is a sequence such that, for some \( p \in X \), \( x_n \in \text{st}(p, G_n) \), then \( \langle x_n \rangle \) has a cluster point in \( (X, T) \); in this case, we say (following Hodel) that \( \langle G_n \rangle \) is a \( w\Delta \)-sequence for \( (X, T) \).
Theorem 1.3. A topological space admits a K-developable semimetric iff it is a wΔ-space with a regular Gδ-diagonal.

Proof. Suppose that d is a K-developable semimetric for (X,T). Let $G_n$ be the set of open sets G in T which have d-diameter less than $2^{-n}$. The set of spheres centered at p is a neighborhood base for p in (X,T); moreover, since d is developable, there are spheres of arbitrarily small diameter centered at p. Consequently, $G_n$ is a cover of X.

Since $\text{st}(p,G_n) \subseteq S_d(p,2^{-n})$, we conclude that $(G_n)$ is a wΔ-sequence. Finally, letting $U_n = \bigcup \{G \times G \mid G \in G_n\}$, we claim that the intersection of the closures of $U_n$ is the diagonal of X. Otherwise, there are distinct p and q such that, for each n, there is $(x_n,y_n) \in G \times G$, for some $G \in G_n$, such that $(x_n,y_n) \in S_d(p,2^{-n}) \times S_d(q,2^{-n})$. But, (X,T) is Hausdorff, since it is K-semimetrizable. Hence, we may choose disjoint open sets U and V with p $\in$ U and q $\in$ V. Now, choose m $\in$ N so that $x_n \in U$ and $y_n \in V$ for all n $\geq$ m. It follows that $\{x_n \mid n \geq m\} \cup \{p\}$ and $\{y_n \mid n \geq m\} \cup \{q\}$ are disjoint compact sets that are not separated by d.

Conversely, suppose that $(w_n)$ is a wΔ-sequence and that $(U_n)$ is a decreasing sequence of open sets in $X \times X$ such that $\Delta_X = \bigcap \{U_n \mid n \in N\} = \bigcap \{U_n \mid n \in N\}$. Let $U_n = \{G \in T \mid G \times G \subseteq U_n\}$; note that $U_n$ is a cover of X. With
appropriate finite intersections of sets from these covers we may construct a sequence \( \langle G_n \rangle \) of open covers such that
\[ G_{n+1} \subseteq G_n, \text{ for each } n, \text{ and } G_n \text{ refines both } \omega_n \text{ and } U_n. \]
(Note that \( \langle G_n \rangle \) is also a \( \omega \)-sequence.)

There is a distance function \( d : X \times X \to \mathbb{R} \) such that, if \( x \neq y \), then \( d(x, y) = 2^{-n} \), where \( n \) is the first positive integer such that \( x \not\in \text{st}(y, G_n) \). Note that \( S_d(p, 2^{-n}) = \text{st}(p, G_n) \) so that \( T_d \subseteq T \). Furthermore, we claim that \( \{ S_d(p, 2^{-n}) \mid n \in \mathbb{N} \} \) is a neighborhood base for \( p \) in \( (X, T) \). Otherwise, obtain \( G \in T \) and a sequence \( \langle a_n \rangle \) such that \( p \in G, a_n \in S_d(p, 2^{-n}), \) but \( a_n \not\in G \). Since \( \langle G_n \rangle \) is a \( \omega \)-sequence, it must be that \( \langle a_n \rangle \) clusters at a point \( q \not= p \) and, since \( \langle G_n \rangle \) refines \( \langle U_n \rangle \), there is \( V \in T \) such that \( q \in V \) and \( V \cap \text{st}(p, G_n) = \emptyset \) for some \( n \). This contradicts that \( q \) is a cluster point of \( \langle a_n \rangle \). Thus, \( d \) is an admissible semimetric for \( (X, T) \); moreover, \( d \) is developable since each open set in \( G_n \) has \( d \)-diameter less than \( 2^{-n} \).

Finally, we show that \( d \) is a \( K \)-semimetric. If \( A \) and \( B \) are compact sets such that \( d[A, B] = 0 \), then choose sequences \( \langle a_n \rangle \) in \( A \) and \( \langle b_n \rangle \) in \( B \) such that \( d(a_n, b_n) \to 0 \), \( d(a_n, p) \to 0 \) and \( d(b_n, q) \to 0 \), for some point \( p \in A \) and some point \( q \in B \). We claim that \( p = q \) which completes the proof. Otherwise, there are \( m \) and \( k \) such that \( S_d(p, 2^{-k}) \times S_d(q, 2^{-k}) \cap U_m = \emptyset \). Now, there are \( a_n \in S_d(p, 2^{-k}) \) and \( b_n \in S_d(q, 2^{-k}) \) such that \( d(a_n, b_n) < 2^{-m} \).
hence, \( a_n \in \text{st}(b_n, G_m) \) which contradicts that
\[
S_d(p, 2^{-k}) \times S_d(q, 2^{-k}) \cap U_m = \emptyset.
\]

Our approach also provides an easy proof to an analogous theorem of Hodel. A topological space \( X \) has a \( G_\delta \)-diagonal \([6]\) iff there is a \( G_\delta \)-diagonal sequence for \( X \), that is, there is a sequence \( \{G_n\} \) of open covers of \( X \) such that \( \{p\} = \bigcap \{\text{st}(p, G_n)\} \) \( n \in \mathbb{N} \). This definition parallels the well known result that \( X \) has a \( G_\delta \)-diagonal iff there is a sequence \( \{G_n\} \) of open covers of \( X \) such that \( \{p\} = \bigcap \{\text{st}(p, G_n)\} \) \( n \in \mathbb{N} \).

**Theorem 1.4.** \([6]\) A Hausdorff space admits a developable semimetric iff it is a \( \omega \Delta \)-space with a \( G_\delta \)-diagonal.

**Proof.** Suppose that \( d \) is a developable semimetric for the Hausdorff space \((X, T)\). If \( G_n \) is the set of open sets in \( T \) which have \( d \)-diameter less than \( 2^{-n} \), then \( \{G_n\} \) is easily a \( \omega \Delta \)-sequence for \((X, T)\) such that \( \text{st}(p, G_n) \subseteq S_d(p, 2^{-n}) \). For \( q \neq p \), there is an open neighborhood \( G \) of \( p \) such that \( q \not\in G \), from which it follows that \( \{G_n\} \) is also a \( G_\delta \)-diagonal sequence. The converse follows easily using the construction of our Theorem 1.3.

2. Developable Semimetrics and 1-Continuous Semimetrics

Any metric for \( X \) is a continuous distance function; any continuous distance function for \( X \) is both developable and 1-continuous. As in the case of metrics, when \( d \) is a 1-continuous distance function for \( X \), \( T_d \) is a topology
for $X$ for which the set \{ $S_d(p, \varepsilon) \mid p \in X, \varepsilon > 0$ \} of spheres is a base; thus, $d$ is a symmetric for $(X,T)$ iff $d$ is a semimetric for $(X,T)$.

**Theorem 2.1.** For any separable space $(X,T)$, if there is a 1-continuous distance function $d$ for $X$ such that $T_d \subseteq T$, then $(X,T)$ is submetrizable (i.e., there is a metric $\rho$ for $X$ such that $T_\rho \subseteq T$).

**Proof.** Suppose that $d$ is a 1-continuous distance function on $(X,T)$ and that $A = \{ a_n \mid n \in \mathbb{N} \}$ is a countable dense subset. For each $n$, there is a pseudometric $\rho_n$ on $(X,T)$ such that $\rho_n(x,y) = \min\{2^{-n}, |d(x,a_n) - d(y,a_n)|\}$. Since $A$ is dense, the pseudometric $\rho = \Sigma\rho_n$ is a metric for $X$, and, since $d$ is 1-continuous on $X \times X$, it follows that $T_\rho \subseteq T_d \subseteq T$.

**Corollary 2.2.** $\psi_R$ is not 1-continuously semimetrizable.

**Proof.** As we have indicated in Remark 1.2, $\psi_R$ does not have a regular $G_\delta$-diagonal. Consequently, $\psi_R$ is not submetrizable. Since $\psi_R$ is separable, we conclude from Theorem 2.1 that any admissible semimetric for $\psi_R$ can not be 1-continuous.

**Theorem 2.3.** If $(X,T)$ is a semimetrizable space with a zero set diagonal, then $(X,T)$ is $K$-semimetrizable.

**Proof.** Suppose that $d$ is a semimetric for $(X,T)$ and $\alpha : X \times X \to [0,1]$ is a continuous function whose zero set is the diagonal. If $d_1$ is the distance function for $X$
with $d_1(x,y) = \min \{a(x,y), a(y,x)\}$, then $d_1$ separates disjoint compact subsets of $(X,T)$. It follows that $d + d_1$ is a K-semimetric for $(X,T)$.

**Corollary 2.4.** If $(X,T)$ is a separable 1-continuously semimetrisable space, then $(X,T)$ is 1-continuously K-semimetrizable.

**Proof.** This follows easily by applying, first, Theorem 2.1 and then, using the construction of our proof of Theorem 2.3.

**Remark 2.5.** Concerning admissible semimetric types

For any $R$, $\Psi_R$ is developable semimetrizable, but not 1-continuously semimetrizable (Corollary 2.2); it is K-semimetrizable, but not K-developable semimetrizable (Theorem 1.1).

Borges' example $B_1$ is 1-continuously K-semimetrizable (from [1] and Corollary 2.4, since it is separable); it is not developable semimetrizable [1].

Burke's example $B_2$ is developable semimetrizable, but not K-semimetrizable [2]; it is not 1-continuously semimetrizable (from Corollary 2.4, since it is separable).

**Remark 2.6.** Concerning $G_\delta$-diagonal types

The Isbell-Mrówka spaces $\Psi_R$ and Burke's example $B_2$ are separable $w\Delta$-spaces which have $G_\delta^*$-diagonals, but do not have regular $G_\delta$-diagonals; see Theorems 1.3 and 1.4.
Borges' example \( B_1 \) is a separable space with a zero set diagonal; it is not a \( w\Delta \)-space.

**Remark 2.7. Concerning the Normal Moore Space Conjecture**

Borges' example is normal because it is regular and Lindelof. It is not continuously semimetrizable because it is not developable semimetrizable. Thus, a normal 1-continuously semimetrizable space need not be continuously semimetrizable. This is of some interest because of its relationship to the Normal Moore Space Conjecture. Note that this result does not require additional set-theoretic axioms.

**References**


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