A GENERALIZATION OF SCATTERED SPACES

by

H. Z. Hdeib and C. M. Pareek
A GENERALIZATION OF SCATTERED SPACES

H. Z. Hdeib and C. M. Pareek

1. Introduction

Scattered spaces have been studied by several authors (see [6], [7], [8], [11], [12], [13], [14], [15], [16] and [17]). Recently, in [11], [15] and [17] some generalizations of scattered spaces have been considered and have been extensively studied. Our interest in this topic was stimulated by some questions in [8] and some of the results obtained in [11], [15] and [18].

In this paper, we introduce the concept of $\omega$-scattered spaces as a natural generalization of the concept of scattered spaces. It is proved that in the class of compact Hausdorff spaces the concept of $\omega$-scatteredness of the space coincides with scatteredness. It is noted that $\omega$-scattered need not be scattered in general. Also, the C-scattered spaces introduced in [15] are not comparable with the $\omega$-scattered spaces. We start out by giving a characterization of $\omega$-scattered spaces. Then, a relationship between $\omega$-scatteredness of the space and scatteredness of some extensions is established. This relationship helps us to prove that Lindelöf $P^*$-spaces are functionally countable and Lindelöf $\omega$-scattered spaces are functionally countable. Later on, we show that for a compact Hausdorff space $X$, (i) $X$ is scattered, (ii) $X$ is $\omega$-scattered and (iii) $X$ is functionally countable are
equivalent. Finally, some product theorem for a class of Lindelöf spaces have been established, and it is proved that a $T_3$, first countable, paracompact, and $\omega$-scattered space is metrizable. The last result improves a result of Wicke and Worrell in [18].

2. Preliminaries

In this section some essential definitions are introduced, notations are explained and some basic facts which are essential in obtaining the main results are stated.

Throughout this paper $X$ denotes a $T_1$ space. The symbol $\omega$ and $c$ denote the cardinal number of integers and reals respectively. The cardinality of any set $A$ is denoted by $|A|$.

**Definition 2.1** [9]. A function $f: X \to Y$ is called barely continuous if, for every non-empty closed $A \subseteq X$, the restriction $f|_A$ has at least one point of continuity.

**Definition 2.2** [8]. A space $X$ is called functionally countable if every continuous real valued function on $X$ has a countable image.

**Definition 2.3** [8]. Given a topological space $(X,T)$, $b(X,T)$ will represent the set $X$ with the topology generated by the $G_\delta$-sets of $(X,T)$. Sometimes $T$ is not mentioned and $bX$ is written instead of $b(X,T)$.

**Definition 2.4** [3]. A space $X$ is called a P-space if the intersection of countably many open sets is open.
Now, we list some known results which will be helpful in obtaining the main results.

Theorem 2.5 [2]. If \(X\) is a regular, Lindelöf, scattered space, then \(bX\) is Lindelöf.

Theorem 2.6 [8]. If \(X\) is a regular, Lindelöf, P-space, then \(X\) is a functionally countable.

Theorem 2.7 [9]. If \(f\) is a barely continuous function from a hereditarily Lindelöf space \(X\) onto a space \(Y\), then \(Y\) is Lindelöf.

Theorem 2.8. If \(X\) is a \(T_2\), Lindelof, P-space, then \(X\) is normal.

3. \(\omega\)-Scattered Spaces

A space \(X\) is called \(\omega\)-scattered if every non-empty subset \(A\) of \(X\) has a point \(x\) and an open neighborhood \(U_x\) of \(x\) in \(X\) such that \(|U_x \cap A| \leq \omega\).

Every scattered space is \(\omega\)-scattered but the converse is not true, because every countable space is \(\omega\)-scattered, while the (countable) set of rationals with the usual topology is not scattered.

A space \(X\) is \(C\)-scattered [15], if every non-empty closed subset \(A\) of \(X\) has a point with a compact neighborhood in \(A\). The following remark shows that \(\omega\)-scattered spaces and \(C\)-scattered space are not comparable.
Remark 3.1. The set of rationals $\mathbb{Q}$ with usual topology is $\omega$-scattered. However, it is not C-scattered since no point of $\mathbb{Q}$ has a compact neighborhood.

The set of reals $\mathbb{R}$ with usual topology is C-scattered (in fact, it is locally compact) but not $\omega$-scattered.

A point $x$ of a space $X$ is called a condensation point of the set $A \subseteq X$ if every neighborhood of the point $x$ contains an uncountable subset of $A$.

Definition 3.2 [4]. A subset $A$ of a space $X$ is called $\omega$-closed if it contains all of its condensation points. The complement of an $\omega$-closed set is called $\omega$-open.

Observe that $A \subseteq X$ is $\omega$-open iff for each $x$ in $A$ there is an open set $U$ in $X$ containing $x$ such that $|U - A| < \omega$.

The next theorem characterizes $\omega$-scattered spaces.

Theorem 3.3. For any space $X$ the following are equivalent:

(i) $X$ is $\omega$-scattered.

(ii) Every nonempty $\omega$-closed subset $A$ of $X$ contains a point $x$ which is not a condensation point.

(iii) There exists a well ordering $\leq$ of $X$ such that for each $x \in X$, the set $A_x = \{y \in X \mid y \leq x\}$ has the property that for each $y \in A_x$ there exists an open set
U_y containing y such that \(|U_y \cap (X - A_x)| \leq \omega, \text{ i.e.,}
for each x \in X, the set A_x is \omega-open.

Proof. (i) + (ii) is obvious.

(ii) + (iii). Let X be a space in which every nonempty closed subset has a point which is not a condensation point. Then X has a point x_1 which is not a condensation point. Now, X - {x_1} is \omega-closed in X and therefore X - {x_1} has a point x_2 which is not a condensation point. Then X - {x_1, x_2} is \omega-closed. Finally, using transfinite induction one can complete the proof.

(iii) + (i). Let A be any nonempty subset of X. Since X is well ordered, A has a first element, say x_0. Now, by the hypothesis A_{x_0} = \{y \in X | y \leq x_0\} is \omega-open. Hence, X is \omega-scattered.

Definition 3.4 [5]. A function f: X \rightarrow Y is called \omega-continuous at x \in X if for every open set V containing f(x) there is an \omega-open set U containing x such that f(U) \subseteq V. If f is \omega-continuous at each point of X, then f is \omega-continuous on X. A function f: X \rightarrow Y is called barely \omega-continuous if for every non-empty closed subset A of X, f|_A has at least one point of \omega-continuity.

The following theorem provides a basic tool to obtain some of the main results.

Theorem 3.5. If (X,T) is a topological space and T_\omega is the topology on X having as a base \{U - C | U \in T and
C is finite or countable), then for any $A \subset X$ the following holds:

(i) $A$ is $\omega$-open if and only if $A$ is open in $(X, T_\omega)$, i.e., $A \in T_\omega$.

(ii) $A$ is $\omega$-closed if and only if $A$ is closed in $(X, T_\omega)$, i.e., $X - A \in T_\omega$.

(iii) $f: (X, T) \to Y$ is $\omega$-continuous if and only if $f: (X, T_\omega) \to Y$ is continuous.

(iv) $f: (X, T) \to Y$ is barely $\omega$-continuous if and only if $f: (X, T_\omega) \to Y$ is barely continuous.

The proof is straightforward.

**Theorem 3.6.** If $(X, T)$ is Lindelöf, then $(X, T_\omega)$ is Lindelöf.

The proof is straightforward, therefore left for the reader.

**Theorem 3.7.** If $f: (X, T) \to Y$ is barely $\omega$-continuous and $(X, T)$ is hereditarily Lindelöf, then $Y$ is Lindelöf.

**Proof.** It follows from theorem 3.6 that $(X, T_\omega)$ is hereditarily Lindelöf. By theorem 3.5, $f: (X, T_\omega) \to Y$ is barely continuous. Hence by theorem 2.7, $Y$ is Lindelöf.

**Theorem 3.8.** $(X, T)$ is $\omega$-scattered if and only if $(X, T_\omega)$ is scattered.

The proof is obvious by the Theorem 3.5.
Definition 3.9 [4]. A space $X$ is called a $P^*$-space if the intersection of countably many open sets is $\omega$-open.

Theorem 3.10. If $(X,T)$ is a $T_2$, Lindelöf $P^*$-space, then $(X,T)$ is functionally countable.

Proof. Suppose $(X,T)$ is a Lindelöf $P^*$-space, then by Theorem 3.6, $(X,T_\omega)$ is Lindelöf. Now, $(X,T_\omega)$ is a $T_2$, Lindelöf $P$-space. Thus, by Theorem 2.8, $(X,T_\omega)$ is normal. Hence, by Theorem 2.6, $(X,T_\omega)$ is functionally countable. Let $f: (X,T_\omega) \to (X,T)$ be the identity function. Then, $f$ is continuous. Since $(X,T_\omega)$ is functionally countable, it is easy to see that $(X,T)$ is functionally countable.

Theorem 3.11. $(X,T)$ is $\omega$-scattered if and only if every function $f$ on $(X,T)$ is barely $\omega$-continuous.

Proof. Suppose $(X,T)$ is $\omega$-scattered. Let $f: (X,T) \to Y$ be a function from $(X,T)$ onto an arbitrary space $Y$. Let $A$ be any $\omega$-closed subset of $X$. Then, $A$ contains a point $x_0$ which is not a condensation point by Theorem 3.3. Now, it is easy to conclude that $f|_A$ is $\omega$-continuous at $x_0$. Hence, $f$ is barely $\omega$-continuous.

For the converse, suppose that any function $f$ from $(X,T)$ onto any space is barely $\omega$-continuous. So, in particular the identity function $i_X$ from $(X,T)$ onto $X$ with discrete topology is barely $\omega$-continuous. Let $A$ be any non-empty $\omega$-closed subset of $X$. Then, $i_X|_A$ is $\omega$-continuous at some $y$ in $A$, i.e., there is an $\omega$-open set $U$ such that $U \cap A = i_X^{-1}(i_X(y)) = \{y\}$. Hence, $(X,T_\omega)$ is scattered. Therefore, by Theorem 3.8 $(X,T)$ is $\omega$-scattered.
Notation. Let $X$ be a topological space. Let $X^{(0)} = X$. Let $X^{(1)}$ denote the collection of condensation points of $X$. With $X^{(\alpha)}$ for an ordinal $\alpha$, let $X^{(\alpha+1)} = (X^{(\alpha)})^{(1)}$. If $\alpha$ is a limit ordinal, let $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}$.

It is easy to see that $X$ is $\omega$-scattered if and only if $X^{(\alpha)} = \emptyset$ for some $\alpha$.

Theorem 3.12. If $X$ is a Lindelöf $\omega$-scattered space then $bX$ is Lindelöf.

Proof. Let $\alpha$ be an ordinal such that $X^{(\alpha)} = \emptyset$. $\alpha$ exists because $X$ is $\omega$-scattered. If $\alpha = 1$, then it is easy to see that $X$ is countable because $X$ is Lindelöf. Hence the result follows. Suppose we have proved the result for all $\beta < \alpha$. That is, if $\beta < \alpha$ and $X^{(\beta)} = \emptyset$, then $bX$ is Lindelöf.

Case 1. There is $\beta < \alpha$ such that $\beta + 1 = \alpha$ and $X^{(\alpha)} = \emptyset$. It is easy to see that $X^{(\beta)}$ is a countable closed subset of $X$. Consider the open cover $U = \{X - X^{(\beta)}\} \cup \{U_x | x \in X^{(\beta)}\}$ where $|U_x \cap X^{(\beta)}| \leq \omega$ for each $x$ and $U_x$ is open in $X$ containing $x$. Since $X$ is regular, there exists an open cover $H$ of $X$ such that the closure of members of $H$ refines $U$. $X$ is Lindelöf implies $H$ has a countable subcover $V$. Now if $V \in V$ and $\overline{V} \subseteq X - X^{(\beta)}$ then $\overline{V^{(\beta)}} = \emptyset$, i.e. $b\overline{V}$ is Lindelöf by the inductive assumption. Let $V' = \{\overline{V} | V \in V', \text{ and } \overline{V} \subseteq (X - X^{(\beta)})\}$. Since $X^{(\beta)}$ is countable we have $bX^{(\beta)}$ is Lindelöf. Now $M = \{X^{(\beta)}\} \cup V'$ is countable closed cover
of $X$ such that for each $M \in M$ we have $bM$ is Lindelöf.
Hence $bX$ is Lindelöf.

Case 2. $\chi^{(\alpha)} = \bigcap_{\beta<\alpha} \chi^{(\beta)} = \emptyset$.

Consider the cover $U = \{X - X^{(\beta)} | \beta < \alpha\}$ of $X$. Since $X$
is regular, there exists an open cover $H$ of $X$ such that
the closures of members of $H$ refines $U$. $X$ is Lindelöf
implies $H$ has a countable subcover $V$. Then for each
$V \in V$, $\overline{V}$ is in some $X - X^{(\beta)}$ for $\beta < \alpha$. Hence, for each
$V \in V$, $\overline{V}^{(\beta)} = \emptyset$. By the inductive assumption, for each
$V \in V$, $b\overline{V}$ is Lindelöf. Therefore, $bX$ is Lindelöf.

**Theorem 3.13.** (i) If $(X,T)$ is a regular, Lindelöf, $\omega$-scattered space, then $(X,T)$ is functionally countable.
(ii) If $X$ is a regular, Lindelöf, $\omega$-scattered space
such that each point of $X$ is a $G_\delta$-set, then $|X| \leq \omega$.

**Proof.** (i) It follows from Theorem 3.12 that $b(X,T)$ is Lindelöf. Also $b(X,T)$ is a $T_2$ $P$-space. Hence by
Theorem 2.6 and 2.8, $b(X,T)$ is functionally countable.
Let $f: b(X,T) \to (X,T)$ be the identity function. Then, $f$
is continuous. Since $b(X,T)$ is functionally countable,
$(X,T)$ is functionally countable.

The proof of (ii) follows easily from the Theorem 3.12.

**Theorem 3.14.** If $(X,T)$ is hereditarily Lindelöf $\omega$-scattered space, then $(X,T)$ is countable.

**Proof.** Suppose $(X,T)$ is hereditarily Lindelöf $\omega$-scattered space. Let $i_X$ be the identity function from
$(X,T)$ into $X$ with discrete topology. Then $i_X$ is barely $ω$-continuous. Hence, by Theorem 3.7, $i_X(X)$ is Lindelöf. Therefore, $X$ is countable.

In [8], the following theorem is attributed to Rudin [13] and Pelczynski and Semadeni [12].

Theorem 3.15. For a compact Hausdorff space the following are equivalent:

(i) $X$ is scattered.

(iii) $X$ is functionally countable.

It is natural to ask whether Theorem 3.15 remains true if we replace scattered by $ω$-scattered. The following theorem gives an affirmative answer to this question.

Theorem 3.16. For a compact Hausdorff space $X$ the following are equivalent:

(i) $X$ is scattered.

(ii) $X$ is $ω$-scattered.

(iii) $X$ is functionally countable.

Proof. (i) $→$ (ii) is obvious

(ii) $→$ (iii). It follows from Theorem 3.13.

(iii) $→$ (i) follows from Theorem 3.15.

4. Product of Lindelöf $ω$-Scattered Spaces

Theorem 4.1. If $bX$ and $Y$ are Lindelöf spaces, then $X \times Y$ is Lindelöf.

The proof that $bX \times Y$ is Lindelöf follows an argument similar to the one used in ([6], Vol. II, page 16).
to prove that the product of two compact spaces is compact. Since \( X \times Y \)'s topology is weaker than \( bX \times Y \)'s, \( X \times Y \) is Lindelöf.

**Theorem 4.2.** If \( X \) is a regular, Lindelöf, \( \omega \)-scattered space and \( Y \) is any Lindelöf space, then \( X \times Y \) is Lindelöf.

**Proof.** It follows from Theorem 3.12 that \( bX \) is Lindelöf. Hence, by Theorem 4.1, \( X \times Y \) is Lindelöf.

**Corollary 4.3.** A finite product of Lindelöf \( \omega \)-scattered spaces is Lindelöf.

In [10], it was shown that a countable product of Lindelöf \( \omega \)-spaces is Lindelöf. Using this result we can obtain the following theorem.

**Theorem 4.4.** A countable product of regular, Lindelöf, \( \omega \)-scattered spaces is Lindelöf.

**Proof.** Let \( \{X_n | n \leq \omega\} \) be a family of Lindelöf \( \omega \)-scattered spaces. Then, by Theorem 3.12, each \( bX_n \) is a Lindelöf. Hence \( \prod_{n\leq \omega} bX_n \) is Lindelöf. Since \( \prod_{n\leq \omega} bX_n \) maps continuously onto \( \prod_{n\leq \omega} X_n \), we obtain that \( \prod_{n\leq \omega} X_n \) is Lindelöf.

In [7], Kunen proved that if each \( X_n \) is a Hausdorff compact scattered space, then the box product \( \Box_{n\leq \omega} X_n \) is c-Lindelöf.

In view of Theorem 3.16, we can state Kunen's result as follows:
Theorem 4.5. If each \( X_n \) is a Hausdorff compact, \( \omega \)-scattered space, then the box product \( n \leq \omega \)
\( n \times \prod_{<\omega} X_n \) is \( c \)-Lindelöf.

5. Metrizability of \( \omega \)-Scattered Spaces

In [18], it was shown that every regular, first countable, paracompact, scattered space is metrizable. In this section, we obtain a generalization of this result using \( \omega \)-scattered spaces.

Definition 5.1 [11]. A space \( X \) is called \( \sigma \)-discrete if it is a union of countably many closed discrete subspaces.

Definition 5.2. A space \( X \) is called \( F_\sigma \)-screenable if every open cover of \( X \) has a \( \sigma \)-discrete closed refinement.

Definition 5.3. A subset \( Y \) of a space \( X \) is called locally countable if for each \( y \in Y \) there is an open neighborhood \( U_y \) in \( X \) containing \( y \) such that \( |U_y \cap Y| \leq \omega \).

Lemma 5.4. If \( X \) is \( F_\sigma \)-screenable (or metalindelöf) and locally countable, then \( X \) is \( \sigma \)-discrete.

Proof. We prove the lemma when \( X \) is \( F_\sigma \)-screenable and locally countable. The other case follows similarly. By the assumptions, \( X \) has an open cover \( U = \{U_\beta \mid \beta \in \Gamma \} \) such that \( |U_\beta| \leq \omega \) for each \( \beta \in \Gamma \). \( X \) is \( F_\sigma \)-screenable implies there exists a \( \sigma \)-discrete closed refinement
\[
F = \bigcup_{i=1}^{\infty} F_i \text{ where } F_i = \{F_i^\alpha \mid \alpha \in \Lambda_i \} \text{ for } i \in \mathbb{N}.
\]
Since
each $U_\beta$ is countable and $F$ refines $U$, we see that
$|F_{i\alpha}| \leq \omega$ for each $i$ and $\alpha$. Hence $F_{i\alpha}$ is $\sigma$-discrete for each $i$ and $\alpha$. Let $F_{i\alpha} = \{x_{ij\alpha} | j \in \mathbb{N}\}$ and $G_{ij} = \{x_{ij\alpha} | \alpha \in \Lambda_i\}$. Then it is obvious that $G_{ij}$ is discrete, closed and $X = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} G_{ij}$. Therefore, $X$ is $\sigma$-discrete.

Lemma 5.5. If $X$ is $F_\sigma$-screenable and $\omega$-scattered, then $X$ is $\sigma$-discrete.

Proof. Let $\alpha$ be an ordinal such that $X^{(\alpha)} = \phi$. $\alpha$ exists because $X$ is $\omega$-scattered. If $\alpha = 1$, then it is easy to see that $X$ is locally countable and by Lemma 5.4 the result follows. Suppose we have proved the result for all $\beta < \alpha$ and $X^{(\beta)} = \phi$, then $X$ is $\sigma$-discrete.

Case 1. There is $\beta < \alpha$ such that $\alpha = \beta + 1$ and $X^{(\alpha)} = \phi$. It is easy to see that $X^{(\beta)}$ is a closed locally countable subset of $X$. Consider the open cover $U = \{X - X^{(\beta)}\} \cup \{U_x | x \in X^{(\beta)}\}$ where $|U_x \cap X^{(\beta)}| \leq \omega$ for each $x$ and $U_x$ is open in $X$ containing $x$. $X$ is $F_\sigma$-screenable implies $U$ has a $\sigma$-discrete closed refinement $V = \bigcup_{n=1}^{\infty} V_n$ where $V_n = \{V_{n\lambda} | \lambda \in \Lambda_n\}$. Note that each $V_{n\lambda}$ is $F_\sigma$-screenable and $\omega$-scattered. Also if $V_{n\lambda} \subseteq X - X^{(\beta)}$ then $V_{n\lambda}^{(\beta)} = \phi$, i.e. $V_{n\lambda}$ is $\sigma$-discrete by the inductive assumption. Let $V' = \{V | V \in V, \text{ and } V \subseteq X - X^{(\beta)}\}$, then $V'$ covers $X - X^{(\beta)}$. Since $X^{(\beta)}$ is a closed subset of $X$, it follows by Lemma 5.4 that $X^{(\beta)}$ is $\sigma$-discrete. Now $W = \{X^{(\beta)}\} \cup V'$ is a $\sigma$-discrete closed cover of $X$ with
each member is \(\sigma\)-discrete. Hence, it is easy to conclude that \(X\) is \(\sigma\)-discrete.

\textit{Case 2.} \(X^{(\alpha)} = \bigcap_{\beta < \alpha} X^{(\beta)} = \emptyset.\)

Consider the cover \(U = \{X - X^{(\beta)} \mid \beta < \alpha\}\) of \(X\). Let \(V\) be a \(\sigma\)-discrete closed refinement of \(U\). Then, each \(V \in V\) is in some \(X - X^{(\beta)}\) for \(\beta < \alpha\). Hence \(V^{(\beta)} = \emptyset\) for each \(V \in V\). Therefore, for each \(V \in V\), \(V\) is \(\sigma\)-discrete by the inductive assumption. Hence, it is easy to conclude that \(X\) is \(\sigma\)-discrete.

\textbf{Theorem 5.6.} \textit{If \(X\) is a regular, first countable, paracompact, \(\omega\)-scattered space, then \(X\) is metrizable.}

\textbf{Proof.} It follows from Lemma 5.5 that \(X\) is \(\sigma\)-discrete. Now, it is well known that a \(\sigma\)-discrete first countable space is developable. Thus \(X\) is developable. Therefore by Bing's metrization theorem (see [1], p. 408), \(X\) is metrizable.

\textbf{Corollary 5.7 [18].} \textit{If \(X\) is a regular, first countable, scattered, paracompact space, then \(X\) is metrizable.}

Finally, we suggest, the following questions.

\textit{Question 5.8.} Which spaces \((X,T)\) have \((X,T_\omega)\) paracompact?

\textit{Question 5.9.} When are regular Lindelöf, \(\omega\)-scattered spaces, scattered?
References

[12] A. Pelczynski and Z. Semadeni, Spaces of continuous functions (III) (Spaces C(Ω) for Ω without perfect sets), Studia Mathematica 18 (1959), 211-222.


Kuwait University

13060 Safat, Kuwait