SPACES OF CONTINUOUS LINEAR FUNCTIONALS: SOMETHING OLD AND SOMETHING NEW

by

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Let $C(X)$ denote the set of all continuous real-valued functions on a completely regular Hausdorff space $X$ and $C^*(X)$ be the set of bounded functions in $C(X)$. Let us denote by $C_k(X)$ (respectively by $C_p(X)$) the set $C(X)$ topologized with the compact-open (respectively the point-open) topology. Both $C_k(X)$ and $C_p(X)$ are locally convex spaces. The locally convex compact-open topology on $C(X)$ is generated by the collection of seminorms $\{p_K: K$ is a compact subset of $X\}$ where $p_K(f) = \sup \{|f(x)|: x \in K\}$ for $f \in C(X)$. Similarly the locally convex point-open topology on $C(X)$ is generated by the collection of seminorms $\{p_F: F$ is a finite subset of $X\}$ where $p_F(f) = \sup \{|f(x)|: x \in F\}$. Let $K(X) = \{K \subseteq X: K$ is a compact subset of $X\}$ and $F(X) = \{F \subseteq X: F$ is a finite subset of $X\}$.

Basic open sets in $C_k(X)$ (respectively in $C_p(X)$) look like $\langle f, A, \varepsilon \rangle = \{g \in C(X): |f(x) - g(x)| < \varepsilon$ for all $x \in A\}$ where $f \in C(X)$, $A \in K(X)$ (respectively $A \in F(X)$) and $\varepsilon > 0$.

Let $A_k(X)$ (respectively $A_p(X)$) be the set of all continuous linear functionals (real-valued functions) on $C_k^*(X)$ (on $C_p^*(X)$ respectively). Note since $C_k^*(X)$ (respectively $C_p^*(X)$) is a dense linear subspace of the
locally convex space $C_k(X)$ (respectively $C_p(X)$), the set of all continuous linear functionals on $C_{k}^{*}(X)$ (respectively $C_{p}^{*}(X)$) equals the set of all continuous linear functionals on $C_k(X)$ (respectively on $C_p(X)$). In [8], a normed linear space whose underlying set is $\Lambda_k(X)$ has been studied in detail. In [8], the notation $\Lambda(X)$ has been used in place of $\Lambda_k(X)$. A necessary condition for this normed linear space $\Lambda_k(X)$ to be complete is that $C(X) = C^{*}(X)$, that is, every real-valued continuous function on $X$ must be bounded. In this paper, we want to put the problem of completeness of $\Lambda_k(X)$ in a proper perspective and we show that the problem of completeness of $\Lambda_k(X)$ is essentially a problem of finding a suitable topology on $C^{*}(X)$. Because of the discussion in this paragraph, from now on, we will be interested only in $C^{*}(X)$. We want to answer the problem of completeness of $\Lambda_k(X)$ in a more general setting. For this purpose, we first define a new topology on $C^{*}(X)$ and we will see that the point-open, compact-open and sup-norm topologies on $C^{*}(X)$ are all special cases of this topology.

1. A New Topology on $C^{*}(X)$

Let $\alpha$ be a collection of subsets of $X$ which satisfies the following two conditions: (i) each member of $\alpha$ is $C^{*}$-embedded and (ii) if $A, B \in \alpha$, then there exists $C \in \alpha$ such that $A \cup B \subseteq C$.

For each $A \in \alpha$, define a seminorm $p_A$ on $C^{*}(X)$ as follows. For $f \in C^{*}(X)$, $p_A(f) = \sup \{|f(x)| : x \in A\}$. 
Consider the locally convex topology on $C^*(X)$ generated by the collection of seminorms $\{p_A: A \in \alpha\}$. Because of (ii), for each $f \in C^*(X)$, $f + U = \{f + V: V \in U\}$ is a neighborhood base at $f$ where $U = \{V_{p_A, \epsilon}: A \in \alpha, \epsilon > 0\}$.

We call this new locally convex topology on $C^*(X)$ $\alpha$-topology and the corresponding topological space we denote by $C^*_\alpha(X)$. Note when $\alpha = K(X)$ or $F(X)$, we get compact-open or point-open topology on $C^*(X)$ respectively.

The supremum norm on $C^*(X)$ is defined as $\|f\|_\infty = \sup \{|f(x)|: x \in X\}$ for $f \in C^*(X)$. This supremum norm generates a finer topology than the $\alpha$-topology on $C^*(X)$. We denote this normed linear space by $C^*_\alpha(X)$. If $\alpha$ contains $X$, then $C^*_\alpha(X) = C^*(X)$; and if, in addition, we assume the members of $\alpha$ to be closed, then $C^*_\alpha(X) = C^*_\infty(X)$ only if $\alpha$ contains $X$. (see [7], page 7).

Let $\Lambda^\alpha_\infty(X)$ be the set of all continuous linear functionals (real-valued) on $C^*_\alpha(X)$ and let $\Lambda^\alpha_\infty(X)$ be the set of all continuous linear functionals (real-valued) on $C^*_\infty(X)$. Since the sup-norm topology on $C^*(X)$ is finer than the $\alpha$-topology on it, $\Lambda^\alpha_\infty(X) \subset \Lambda^\alpha_\infty(X)$. Now $\Lambda^\alpha_\infty(X)$ is a normed linear space with the usual conjugate norm, that is, given $\lambda \in \Lambda^\alpha_\infty(X)$, we have a norm $\|\lambda\|_\ast = \sup \{|\lambda(f)|: f \in C^*(X), \|f\|_\infty \leq 1\}$ where $\|\cdot\|_\ast$ is the sup-norm on $C^*(X)$. Consequently we can assign this $\|\cdot\|_\ast$-norm on $\Lambda^\alpha_\infty(X)$ to make it a normed linear space $(\Lambda^\alpha_\infty(X), \|\cdot\|_\ast)$.

Note $\Lambda^\alpha_\infty(X)$ is actually a particular case of $\Lambda^\alpha_\infty(X)$. Here we also mention another particular $\Lambda^\alpha_\infty(X)$. Let $X$ be
a normal Hausdorff space and \(\sigma = \{\text{cl}_X A : A \text{ is a }\sigma\text{-compact subset of } X\}\). Note that \(\sigma\) is closed under finite union because \(\bigcup_{n=1}^k \text{cl}_X A_n = \text{cl}_X \bigcup_{n=1}^k A_n\). We denote the corresponding \(\Lambda_\sigma(X)\) by \(\Lambda_\sigma(X)\). While considering \(\Lambda_\sigma(X)\), we will always assume \(X\) to be a normal Hausdorff space.

2. Basic Properties of \(\Lambda_\alpha(X)\)

Let \(\Lambda_\alpha^+(X) = \{\lambda \in \Lambda_\alpha(X) : \lambda \geq 0\}\) where \(\lambda \geq 0\) provided that \(\lambda(f) \geq 0\) for each \(f \in C^*(X)\) such that \(f \geq 0\). If \(\lambda \in \Lambda_\alpha(X)\) and \(A\) is a subset of \(X\), then \(\lambda\) is said to be supported on \(A\) provided that whenever \(f \in C^*(X)\) with \(f|_A = 0\), then \(\lambda(f) = 0\). Since \(\lambda\) is linear, this is equivalent to saying that whenever \(f, g \in C^*(X)\) with \(f|_A = g|_A\), then \(\lambda(f) = \lambda(g)\).

The next two lemmas can be proved in manners similar to Lemmas 1.1 and 1.2 in [8].

Lemma 2.1. For each \(\lambda \in \Lambda_\alpha(X)\), there exists an element \(A\) in \(\sigma\) such that \(\lambda\) is supported on \(A\). Conversely, if \(\lambda\) is a positive linear functional on \(C^*(X)\) which is supported on an element of \(\sigma\), then \(\lambda \in \Lambda_\alpha^+(X)\).

Lemma 2.2. Let \(A\) be a closed subset of \(X\), let \(F \in \sigma\) and let \(\lambda \in \Lambda_\alpha(X)\). If \(\lambda\) is supported on each of \(A\) and \(F\), then \(\lambda\) is supported on \(A \cap F\).

Now on \(\Lambda_\alpha^+(X)\) we give a topology induced by the metric \(d_*(\lambda, \mu) = \|\lambda - \mu\|_*\) for \(\lambda, \mu \in \Lambda_\alpha^+(X)\).
Theorem 2.3. \((\Lambda^+\alpha(X), d_\star)\) is a closed subspace of 
\((\Lambda\alpha(X), \|\cdot\|_\star)\).

Proof. Let \(\lambda \in \Lambda\alpha(X) \setminus \Lambda^+\alpha(X)\). Then there exists a 
g \in \mathcal{C}(X)\) such that \(g \geq 0\) and \(\lambda(g) < 0\). Let \(r\) be a posi­
tive number such that \(||rg||_\omega < 1\). Define \(\varepsilon = -\frac{r}{2}\lambda(g)\). Now 
suppose \(u \in \Lambda\alpha(X)\) is such that \(||u - \lambda||_\star < \varepsilon\). Then 
\(|u(rg) - \lambda(rg)| < \varepsilon\) so that \(u(rg) - \lambda(rg) < \frac{\varepsilon}{r} = -\frac{1}{2}\lambda(g)\). 
Therefore \(u(g) < \frac{1}{2}\lambda(g) < 0\) so that \(u \in \Lambda\alpha(X) \setminus \Lambda^+\alpha(X)\).

3. The completeness of \(\Lambda^\alpha(X)\) and \(\Lambda\alpha(X)\)

The space \(\Lambda^\alpha(X)\) is a metric space with the metric \(d_\star\). 
This space is complete provided that if a sequence in 
\(\Lambda^\alpha(X)\) is a Cauchy sequence with respect to \(d_\star\), then it 
converges. Likewise the normed linear space \(\Lambda\alpha(X)\) is 
complete if it is complete with respect to its norm \(\|\cdot\|_\star\), 
that is, if it is a Banach space.

We have studied the completeness of \(\Lambda^\kappa(X)\) and \(\Lambda^\alpha(X)\) 
in [8]. We already know that \(\Lambda\omega(X)\), being the conjugate 
space of a normed linear space, is always complete.

To establish that the completeness of \(\Lambda^\alpha(X)\) is 
equivalent to the completeness of \(\Lambda\alpha(X)\), we need the fol­
lowing theorem which can be proved like Theorem 2.2 in [8].

Theorem 3.1. Each \(\lambda \in \Lambda\alpha(X)\) can be written as 
\(\lambda = \lambda^+ - \lambda^-\) where \(\lambda^+\) and \(\lambda^-\) are members of \(\Lambda^\alpha(X)\). Furthermore, if \(\lambda, \mu \in \Lambda\alpha(X)\), then 
\(\|\lambda^+ - \mu^+\|_\star \leq \|\lambda - \mu\|_\star\) and 
\(\|\lambda^- - \mu^-\|_\star \leq \|\lambda - \mu\|_\star\).
Theorem 3.2. The metric space $\Lambda^+_\alpha(X)$ is complete if and only if the normed linear space $\Lambda^+_\alpha(X)$ is complete.

Proof. Use Theorems 3.1 and 2.3.

Because of Theorem 3.2, each of the following theorems about $\Lambda^+_\alpha(X)$ is also true for $\Lambda^+_\alpha(X)$.

Theorem 3.3. Suppose $X$ is infinite and $F(X) \subseteq \alpha$. Now if $\Lambda^+_\alpha(X)$ is complete, then every countable subset of $X$ is contained in some member of $\alpha$.

Proof. Let $A = \{x_n : n \in \mathbb{N}\}$ be any countable subset of $X$. For each $m \in \mathbb{N}$, define $\lambda_m : C^*_\alpha(X) \rightarrow \mathbb{R}$ as follows. For each $f \in C^*_\alpha(X)$, take $\lambda_m(f) = \sum_{n=1}^{m} \frac{1}{2^n} f(x_n)$. Each $\lambda_m$ is a positive linear functional on $C^*_\alpha(X)$ supported on the finite set $\{x_1, \ldots, x_m\}$. Then by Lemma 2.1, $\lambda_m$ is continuous. Now for each $k$ and $m$ with $k < m$, $d_\alpha(\lambda_k, \lambda_m) = \sum_{n=k+1}^{m} \frac{1}{2^n}$. Therefore $(\lambda_m)$ is a Cauchy sequence in $\Lambda^+_\alpha(X)$. Since $\Lambda^+_\alpha(X)$ is complete, the $(\lambda_m)$ converges to some $\lambda$ in $\Lambda^+_\alpha(X)$. Also $\lambda_m + \lambda$ implies $\lambda(f) = \lim_{m \to \infty} \lambda_m(f) = \sum_{n=1}^{\infty} \frac{1}{2^n} f(x_n)$ for all $f \in C^*_\alpha(X)$.

Now suppose $\lambda$ has a support $Y$ which belongs to $\alpha$. We show that $A \subseteq Y$. Suppose not, then there is some $m$ such that $x_m \notin Y$. Since $X$ is completely regular, there is some continuous function $f$ on $X$ with values in the unit interval $I$ such that $f(x_m) = 1$ and $f(Y) = \{0\}$. Since $\lambda$ is supported on $Y$, $\lambda(f) = 0$. But $\lambda(f) = \sum_{n=1}^{\infty} \frac{1}{2^n} f(x_n) \geq \frac{1}{2^m} f(x_m) = \frac{1}{2^m} > 0$. With this contradiction, it follows that $A \subseteq Y$.

Corollary 3.4. If $X$ is infinite, then $\Lambda^+_p(X)$ and $\Lambda^+_p(X)$ are not complete.
Theorem 3.5. If the closure of each countable union of elements of \( \alpha \) belongs to \( \Lambda_\alpha^+(X) \) is complete.

Proof. Let \( (\lambda_n) \) be a Cauchy sequence in \( \Lambda_\alpha^+(X) \). Consider \( \Lambda_\alpha^+(X) \) as a subspace of the complete metric space \( \Lambda_\alpha^+(X) \). Then \( (\lambda_n) \) is a Cauchy sequence in \( \Lambda_\alpha^+(X) \) and hence converges to some \( \lambda \) in \( \Lambda_\alpha^+(X) \). Suppose each \( \lambda_n \) is supported on \( A_n \) where \( A_n \in \alpha \). We show that \( \lambda \) is supported on \( A = \text{cl}_X(\bigcup_{n=1}^{\infty} A_n) \). Let \( f \in C^*(X) \) with \( f|_A = 0 \). Since each \( \lambda_n \) is supported on \( A_n \subseteq A \), then each \( \lambda_n(f) = 0 \) and consequently \( \lambda(f) = \lim_{n \to \infty} \lambda_n(f) = 0 \). Therefore \( \lambda \) has support \( A \). But by hypothesis \( A \in \alpha \). Hence by Lemma 2.1 \( \lambda \in \Lambda_\alpha^+(X) \). So \( \Lambda_\alpha^+(X) \) is complete.

Corollary 3.6. Suppose \( X \) is a normal Hausdorff space. Then \( \Lambda_\alpha^+(X) \) is always complete.

Proof. Suppose for each \( n \), \( A_n \) is a \( \sigma \)-compact subset of \( X \). Then \( \text{cl}_X(\bigcup_{n=1}^{\infty} \text{cl}_X A_n) = \text{cl}_X(\bigcup_{n=1}^{\infty} A_n) \in \sigma \).

4. Measure Theoretic-Counterparts

In this section, we will talk about the measure-theoretic counterparts of \( \Lambda_\alpha(X) \) and \( \Lambda_\infty(X) \) with some extra conditions on \( \alpha \) and \( X \). So now we introduce some ideas from measure theory.

The algebra generated by the closed sets of \( X \) are denoted by \( A_C \) while the \( \sigma \)-algebra they generate is denoted by \( \mathcal{B} \), called the Borel sets.

For us a finitely additive measure (also called signed measure) on \( A_C \) is a real-valued function defined
on $A_c$ satisfying the following two properties (i) $\mu(\emptyset) = 0$; (ii) $\mu(A \cup B) = \mu(A) + \mu(B)$ if $A, B \in A_c$ and $A \cap B = \emptyset$.

A finitely additive measure $\mu$ is called a countably additive measure or simply a measure provided that (iii) $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ for all pairwise disjoint sequences $(A_n)_{n=1}^{\infty}$ such that $A_n \in A_c$ and $\bigcup_{n=1}^{\infty} A_n \in A_c$. When a measure $\mu$ is defined on $\mathcal{B}$, we call it a Borel measure. A measure $\mu$ defined on $\mathcal{B}$ has support $A$ where $A \subseteq X$ and $A \in \mathcal{B}$ if $|\mu|(X \setminus A) = 0$. A finitely additive measure $\mu$ defined on $A_c$ or $\mathcal{B}$ is regular whenever $A$ is in the domain of definition of $\mu$ and $\varepsilon > 0$, there are closed and open sets $C$ and $U$ such that $C \subseteq A \subseteq U$ and $|\mu|(U \setminus C) < \varepsilon$.

Note when $\mu$ has compact support, this definition of regularity coincides with the one usually given in the books on measure theory. For more information on measure theory see [4] and [6].

Now we fix some notations.

A (signed) measure $\mu$ defined on $\mathcal{B}$ is said to be a finite (signed) measure if $|\mu(A)| < \infty$ holds for each $A \in \mathcal{B}$. It can be shown that a signed measure $\mu$ is finite if and only if $|\mu|(X) < \infty$. So a finite signed measure defined on $\mathcal{B}$, has finite total variation. For details on the above, see [1], 26.

Now let $M_+(X)$ be the set of all finite (signed) regular Borel measures on $X$. Let $M_+(X) = \{\mu \in M_+(X) : \mu \geq 0\}$, that is, $\mu$ is a positive measure. Throughout the remaining part of this paper we will assume the following extra condition on $\alpha$: the members of $\alpha$ are closed.
Now define \( M_{\mathcal{B},a}(X) = \{ \mu \in M_{\mathcal{B}}(X): \mu \) has a support \( A(\subset X) \) such that \( A \in a \} \). Let \( M^+_{\mathcal{B},a}(X) = \{ \mu \in M_{\mathcal{B},a}(X): \mu \geq 0 \} \). When \( a = K(X) \) or \( F(X) \), we write \( M_{\mathcal{B},K}(X) \) or \( M_{\mathcal{B},F}(X) \) respectively.

The next thing to observe is that given \( \mu \in M_{\mathcal{B}}(X) \),\\( ||\mu|| = |\mu|(X) \) defines a norm on \( M_{\mathcal{B}}(X) \). So \( (M_{\mathcal{B}}(X),||\cdot||) \) is actually a normed linear space. Also \( M^+_{\mathcal{B}}(X) \) is a metric space when equipped with the norm \( p \) given by \( p(\mu_1,\mu_2) = ||\mu_1 - \mu_2|| \) for every \( \mu_1,\mu_2 \in M^+_{\mathcal{B}}(X) \). Note \( (M_{\mathcal{B},a}(X),||\cdot||) \) is a normed linear space while \( (M^+_{\mathcal{B},a}(X),p) \) is a metric space.

Before having our first theorem in this section, we need the following two lemmas.

**Lemma 4.1.** Suppose \( Y \) is a Borel subset of a completely regular Hausdorff space \( X \). Let \( \mathcal{B}(X) \) and \( \mathcal{B}(Y) \) be the \( \sigma \)-algebras of Borel subsets of \( X \) and \( Y \) respectively. Then \( \mathcal{B}(X) \cap Y = \mathcal{B}(Y) \) where \( \mathcal{B}(X) \cap Y = \{ B \cap Y: B \in \mathcal{B}(X) \} \).

**Proof.** Define \( \mathcal{D} = \{ A \in P(X): A = E \cup (B\setminus Y); E \in \mathcal{B}(Y) \) and \( B \in \mathcal{B}(X) \} \) where \( P(X) \) is the power set of \( X \). Note \( X\setminus (E \cup (B\setminus Y)) = (Y\setminus E) \cup ((X\setminus (B\setminus Y))\setminus Y) \). Now it can be easily shown that \( \mathcal{D} \) is a \( \sigma \)-algebra on \( X \) containing all the closed subsets of \( X \). Hence \( \mathcal{B}(X) \subseteq \mathcal{D} \). So \( \mathcal{B}(X) \cap Y \subseteq \mathcal{D} \cap Y \). But \( \mathcal{D} \cap Y = \mathcal{B}(Y) \). So \( \mathcal{B}(X) \cap Y \subseteq \mathcal{B}(Y) \). Note \( \mathcal{B}(X) \cap Y \) is a \( \sigma \)-algebra on \( Y \) and if \( C \) is a closed subset of \( Y \), then \( C = C' \cap Y \) for some closed subset \( C' \) of \( X \) which means \( C \in \mathcal{B}(X) \cap Y \). Hence \( \mathcal{B}(Y) \subseteq \mathcal{B}(X) \cap Y \). Therefore \( \mathcal{B}(X) \cap Y = \mathcal{B}(Y) \).
Lemma 4.2. If \( A \) is a compact subset of a completely regular Hausdorff space \( X \), then for every closed set \( B \subseteq X \setminus A \), there exists a continuous function \( f: X \to I \) such that \( f(x) = 0 \) for \( x \in A \) and \( f(x) = 1 \) for \( x \in B \).

Proof. See [5], page 168.

Theorem 4.3. Suppose \( \alpha \subseteq K(X) \), that is, the members of \( \alpha \) are compact. Then \( (M_{b,\alpha}(X), \| \cdot \|) \) is isometrically isomorphic to \( (\Lambda_{\alpha}(X), \| \cdot \|_{\ast}) \) while \( M_{b,\alpha}^{+}(X) \) is identified with \( \Lambda_{\alpha}^{+}(X) \) under this isometric isomorphism.

Proof. Define \( F: M_{b,\alpha}(X) \to \Lambda_{\alpha}(X) \) by \( F(\mu)(f) = \int f \, d\mu \) for each \( \mu \in M_{b,\alpha}(X) \) and \( f \in C_{\alpha}^{*}(X) \). Let \( K \) be a compact support of \( \mu \) belonging to \( \alpha \), that is, \( |\mu|(X \setminus K) = 0 \) and \( K \in \alpha \). Then for each \( f \in C_{\alpha}^{*}(X) \), \( |F(\mu)(f)| = |\int f \, d\mu| = |\int_{K} f \, d\mu| \leq |\int_{K} f \, d\mu| \leq |\mu|(X) \cdot P_{K}(f) \) and so \( F(\mu) \) is continuous. Clearly \( F(\mu) \) is linear. Hence \( F(\mu) \in \Lambda_{\alpha}(X) \).

Also \( \| F(\mu) \|_{\ast} \leq \sup \{ |\mu|(K) \cdot P_{K}(f) : f \in C_{\alpha}^{*}(X), \| f \|_{\infty} \leq 1 \} = |\mu|(K) = \| \mu \|_{\ast} \).

Now we prove the reverse inequality, that is, \( \| \mu \|_{\ast} \leq \| F(\mu) \|_{\ast} \).

Note \( |\mu|(K) = \sup \{ \Sigma |\mu|(A_{i}) : \{ A_{i} \} \) is a finite disjoint collection of \( B \) with \( \bigcup A_{i} \subseteq K \}. \) So given \( \epsilon > 0 \), there exist \( A_{1}, \ldots, A_{n} \in B \) such that \( A_{i}'s \) are pairwise disjoint and \( \Sigma_{i=1}^{n} |\mu|(A_{i}) > |\mu|(K) - \epsilon \). Since \( \mu \) is regular there exist compact sets \( C_{i} \) and open sets \( U_{i} \) such that \( C_{i} \subseteq A_{i} \subseteq U_{i} \) and \( |\mu|(U_{i} \setminus C_{i}) < \epsilon/n \) for \( 1 \leq i \leq n \). Since the compact subsets \( C_{i}'s \) are pairwise disjoint, pairwise disjoint open sets \( V_{i} \) exist such that \( C_{i} \subseteq V_{i} \). Now let
\( W_i = U_i \cap V_i \). Then \( C \cap (X \setminus W_i) = \emptyset \). Hence by Lemma 4.2, there exists a continuous function \( f_i : X + I \) such that
\( f_i(C_i) = \{1\} \) and \( f_i(X \setminus W_i) = 0 \). Let \( a_i = \frac{\mu(A_i)}{\mu(A_i)} \) if \( \mu(A_i) \neq 0 \) and if \( |\mu(A_i)| = 0 \), let \( a_i = 0 \). Let \( f = \sum_{i=1}^{n} a_i f_i \). Since \( W_i \)'s are pairwise disjoint, \( \|f\|_{\infty} < 1 \).

Now \( \int f \, d\mu - \sum_{i=1}^{n} |\mu(A_i)| \)
\[ = \left| \sum_{i=1}^{n} a_i \int f_i \, d\mu - \sum_{i=1}^{n} |\mu(A_i)| \right| \]
\[ = \left| \sum_{i=1}^{n} a_i \int f_i \, d\mu - \sum_{i=1}^{n} |\mu(A_i)| \right| \]
\[ = \left| \sum_{i=1}^{n} [a_i f_i \, d\mu - |\mu(A_i)|] + \sum_{i=1}^{n} a_i \int f_i \, d\mu \right| \]
\[ \leq \sum_{i=1}^{n} |a_i| |\mu(C_i) - \mu(A_i)| + \sum_{i=1}^{n} a_i \int f_i \, d\mu \]
\[ \leq \sum_{i=1}^{n} |a_i| |\mu(C_i) - \mu(A_i)| + \sum_{i=1}^{n} a_i \int f_i \, d\mu \]
\[ \leq \sum_{i=1}^{n} |\mu(A_i - C_i)| + \sum_{i=1}^{n} |\mu(W_i - C_i)| \]
\[ < n \cdot \varepsilon + n \cdot \varepsilon = 2\varepsilon. \]

So \( \|F(\mu)\|_{*} \geq \|f \, d\mu\| - \sum_{i=1}^{n} |\mu(A_i)| - 2\varepsilon > |\mu|(X) - 3\varepsilon = |\mu| - 3\varepsilon. \) Therefore \( |\mu| - 3\varepsilon < \|F(\mu)\|_{*} \leq |\mu|. \) Hence \( \|F(\mu)\|_{*} = |\mu|, \) that is, \( F \) is an isometry.

Now we need to show that \( F \) is onto. Suppose \( \lambda \in A_\alpha(X) \). Then \( \lambda \) can be written as \( \lambda = \lambda^+ - \lambda^- \) where
\[ \lambda^+, \lambda^- \in \Lambda^+_\alpha(X). \] Now if \( \lambda \) has a compact support \( K \) belonging to \( \alpha \), then both \( \lambda^+ \) and \( \lambda^- \) have compact support \( K \). To show \( F \) is onto, we try to get \( \mu_1, \mu_2 \in M^+_{\mathbb{D}, \alpha}(X) \) such that 
\[ \lambda^+ = F(\mu_1) \quad \text{and} \quad \lambda^- = F(\mu_2). \]
So \( \lambda = \lambda^+ - \lambda^- = F(\mu_1) - F(\mu_2) = F(\mu_1 - \mu_2) = F(\mu) \) where \( \mu = \mu_1 - \mu_2 \in M^+_{\mathbb{D}, \alpha}(X) \). So we just need to consider \( \lambda^+ \). Define \( \lambda^+_K : C^*(K) \to \mathbb{R} \) as follows. For each \( f \in C^*(K) \), choose an \( f_K \in C^*_\alpha(X) \) such that \( f_K|_K = f \). Then define \( \lambda^+_K(f) = \lambda^+(f_K) \). Since \( \lambda^+ \) is supported on \( K \), \( \lambda^+_K \) is well-defined. Also since \( \lambda^+ \) is linear, so is \( \lambda^+_K \). Finally \( \lambda^+_K \) is continuous since 
\[ \sup \{ |\lambda^+_K(f)| : f \in C^*(K), \|f\|_* \leq 1 \} = \sup \{ |\lambda^+(f)| : f \in C^*(X), \|f\|_* \leq 1 \} = \|\lambda^+\|_* < \infty. \]
By the Riesz Representation Theorem (see [1]), there exists a \( \mu_K \in M^+_{\mathbb{D}}(K) \) such that 
\[ \lambda^+_K(f) = \int_K f \, d\mu_K \text{ for all } f \in C^*(K). \]
It only remains to show that an element \( \mu_1 \in M^+_{\mathbb{D}}(X) \) can be found such that \( \mu_1(B) = \mu_K(B \cap K) \) for all \( B \in \mathcal{B} \).

Then \( \mu_1 \) would be supported on \( K \) so that \( \mu_1 \) would be in 
\[ M^+_{\mathbb{D}, \alpha}(X) \] and thus for each \( f \in C^*(X), \lambda^+(f) = \lambda^+_K(f|_K) = \int_K f|_K \, d\mu_K = \int f \, d\mu_1 = F(\mu_1)(f) \) which shows that \( \lambda^+ = F(\mu_1) \).

First observe that because of Lemma 4.1, \( \mu_1 \) is well-defined on \( \mathcal{B} \). So we only need to show that \( \mu_1 \) is regular. Let \( B \in \mathcal{B} \) and let \( \varepsilon > 0 \). Since \( \mu_K \) is regular, there exists a compact subset \( C \) of \( K \) and an open subset \( U \) of \( K \) such that \( C \subseteq B \cap K \subseteq U \) and \( \mu_K(U \setminus C) < \varepsilon \). Let \( V = U \cup (X \setminus K) \) which is open in \( X \). Then \( C \subseteq B \subseteq V \) and \( \mu_1(V \setminus C) = \mu_K((V \setminus C) \cap K) = \mu_K(U \setminus C) < \varepsilon \). Therefore \( \mu_1 \) is regular and is thus an element of \( M^+_{\mathbb{D}, \alpha}(X) \).
Note when $\alpha = F(X)$ or $K(X)$, the above theorem tells us what is exactly the measure-theoretic counterpart of $\Lambda_p(X)$ or of $\Lambda_k(X)$ respectively. Note that when $\alpha = F(X)$, $M_{b,\alpha}(X)$ is actually the linear space over $\mathbb{R}$ generated by the set of Dirac's measures on $X$. This fact explains why $\Lambda_p(X)$ and $\Lambda^+_p(X)$ cannot be complete because a limit of a Cauchy sequence in $M_{b,\alpha}(X)$ or in $M_{b,\alpha}^+(X)$ may converge to a regular Borel measure on $X$ with infinite support.

Now what is the measure-theoretic counterpart of $\Lambda_\omega(X)$? To answer this question, we introduce a new measure space. Let $M_c(X)$ be the set of all bounded finitely additive regular measures defined on $\mathcal{A}_c$. Again $M_c(X)$ is a normed linear space with the total variation norm. Let $M_c^+(X) = \{\mu \in M_c(X): \mu > 0\}$.

**Theorem 4.4.** If $X$ is a normal and Hausdorff, then $(M_c(X), \|\cdot\|)$ is isometrically isomorphic to $(\Lambda_\omega(X), \|\cdot\|_*)$ while $M_c^+(X)$ is identified with $\Lambda_\omega^+(X)$ under this isometric isomorphism.

**Proof.** See [3], pages 78-83.

But what about the countable additivity of elements of $M_c(X)$? When $X$ is countably compact, we have the following answer.

**Theorem 4.4.** If $X$ is countably compact and if $\mu$ is a bounded regular finitely additive measure defined on $\mathcal{A}_c$, then $\mu$ is countably additive on $\mathcal{A}_c$, that is, $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ whenever $(A_n)$ is a countable family of pairwise
disjoint sets from $A_c$ with union in $A_c$. Moreover $\mu$ has a regular countably additive extension to the $\sigma$-algebra $B$ of Borel subsets of $X$.

Proof. See Theorem 3.11 in [7]. Also see [3].

Now the last theorem can be used to improve Theorem 4.4 to the following version.

Theorem 4.6. If $X$ is countably compact, normal and Hausdorff, then $M^+_B(X)$ is isometrically isomorphic to $\Lambda^+_\infty(X)$ while $M^+_B(X)$ is identified with $\Lambda^+_\infty(X)$ under this isometric isomorphism.

Since $\Lambda^+_\alpha(X) \subseteq \Lambda^+_\infty(X)$, for a countably compact, normal Hausdorff space, we have the following measure-theoretic counterpart of $\Lambda^+_\alpha(X)$.

Theorem 4.7. If $X$ is countably compact, normal and Hausdorff, then $M^+_{B,\alpha}(X)$ is isometrically isomorphic to $\Lambda^+_\alpha(X)$ while $M^+_{B,\alpha}(X)$ is identified with $\Lambda^+_\alpha(X)$ under this isometric isomorphism.

5. Density

The density $d(X)$ of a space $X$ is the smallest infinite cardinal number $m$ such that $X$ has a dense subset which has cardinality less than or equal to $m$. Now a space $X$ is separable if and only if $d(X) = \aleph_0$. If $X$ is a subspace of a metrizable space $Y$, then $d(X) \leq d(Y)$.

Theorem 5.1. For each space $X$, $d(\Lambda^+_\alpha(X)) = d(\Lambda^+_\alpha(X))$.

Proof. Use Theorem 3.1.
Corollary 5.2. $\Lambda_\alpha^+(X)$ is separable if an only if $\Lambda_\alpha(X)$ is separable.

For each $x \in X$, define the evaluation function at $x$, $\phi_x : \mathcal{C}_\alpha^*(X) \rightarrow \mathbb{R}$ by taking $\phi_x(f) = f(x)$ for each $f \in \mathcal{C}_\alpha^*(X)$. Now $\phi_x$ is a positive linear functional on $\mathcal{C}_\alpha^*(X)$ which is supported on $\{x\}$. Now if $\{x\} \in \alpha$, then by Lemma 2.1 $\phi_x \in \Lambda_\alpha^+(X)$.

For the remainder of this section, the notation $|X|$ stands for the cardinality of $X$.

Theorem 5.3. Suppose $F(X) \subseteq \alpha$. Then $|X| \leq d(\Lambda_\alpha^+(X))$.

Proof. Since $F(X) \subseteq \alpha$, $\phi_x \in \Lambda_\alpha^+(X)$ for all $x \in X$. Define the evaluation function $\phi : X \rightarrow \Lambda_\alpha^+(X)$ by taking $\phi(x) = \phi_x$. Since $\mathcal{C}(X)$ separate points, then $\phi$ is one-to-one. Therefore $|\phi(X)| = |X|$. Now let $x$ and $y$ be distinct points of $X$. Then $d_*(\phi_x, \phi_y) = \|\phi_x - \phi_y\|_* = \sup \{|\phi_x(f) - \phi_y(f)| : f \in \mathcal{C}(X), \|f\|_\infty \leq 1\} = \sup \{|f(x) - f(y)| : f \in \mathcal{C}(X), \|f\|_\infty \leq 1\} \geq 1$. So $\phi(X)$ is a discrete subset of $\Lambda_\alpha^+(X)$ and hence $|\phi(X)| \leq d(\phi(X))$. Therefore $|X| = |\phi(X)| \leq d(\phi(X)) \leq d(\Lambda_\alpha^+(X))$.

Corollary 5.4. Suppose $F(X) \subseteq \alpha$ and $\Lambda_\alpha^+(X)$ is separable. Then $X$ is countable.

In order to establish a more general theorem on separability of $\Lambda_\alpha^+(X)$, we need to discuss the separability of $M^+_{b,\alpha}(X)$. Note that the proof of Theorem 3.3 in [8] actually shows that if $X$ is countable, then $M^+_{b}(X)$ is
separable. So when $X$ is countable, $M_{d,\alpha}(X)$ and $M^+_{d,\alpha}(X)$
are also separable.

Theorem 5.5. Suppose $F(X) \subseteq \alpha \subseteq K(X)$. Then $\Lambda^+_{\alpha}(X)$
is separable if and only if $X$ is countable.

Proof. Suppose $\Lambda^+_{\alpha}(X)$ is separable. Then by

Corollary 5.4, $X$ is countable. Conversely, let $X$ be

countable. Now since $\alpha \subseteq K(X)$, by Theorem 4.3, $\Lambda^+_{\alpha}(X)$ is

isomorphic to $M^+_{d,\alpha}(X)$. So $d(\Lambda^+_{\alpha}(X)) = d(M^+_{d,\alpha}(X))$. Hence

$\Lambda^+_{\alpha}(X)$ is separable.

Lastly, we talk about the separability of $\Lambda_{\infty}(X)$. 

Note that Theorem 5.1 gives us $d(\Lambda^+_{\infty}(X)) = d(\Lambda_{\infty}(X))$. This

means that $\Lambda_{\infty}(X)$ is separable if and only if $\Lambda^+_{\alpha}(X)$ is

separable.

Theorem 5.6. $\Lambda_{\infty}(X)$ is separable if and only if $X$ is

compact and countable.

Proof. If $\Lambda_{\infty}(X)$ is separable, then $\Lambda_{k}(X)$ is separable

and so $X$ is countable. Again, since $\Lambda_{\infty}(X)$ is the conjugate

space of the normed linear space $C^*(X)$, $C^*_{\infty}(X)$ is separable.

But this implies that $X$ is compact (see [9], page 54).

Conversely, let $X$ be compact and countable. Since $X$ is

compact, $C^*_{K}(X) = C^*_{K}(X)$ and consequently $\Lambda_{\infty}(X) = \Lambda_{k}(X)$.

But $X$ is countable and so $\Lambda_{k}(X)$ is separable. Hence

$\Lambda_{\infty}(X)$ is separable.

References
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