QUASI-DEVELOPABLE MANIFOLDS

by

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In [RZ] Reed and Zenor showed that a connected, locally connected, locally compact normal Moore space is metrizable. This result re-opened interest in the general question of metrization of manifolds, pending the solution of Wilder's Problem ([RZ], [R]).

Recall that a manifold is a connected regular $T_1$-space for which there is a natural number $n$ such that each point has a neighborhood that is homeomorphic to $\mathbb{R}^n$. Hence manifolds are locally compact and locally connected, but not necessarily metrizable or, equivalently, paracompact. The Reed-Zenor theorem has as a corollary that normal Moore manifolds are metrizable.

For an excellent source of information on non-metrizable manifolds see Peter Nyikos' article in [Nyl].

A natural generalization of a developable space is a quasi-developable space. Recall that a space $X$ is developable (quasi-developable) if there exists a sequence $(G_n : n \in \omega)$ of open covers of $X$ (collections of open subsets of $X$) such that for each $x \in X$, if $U$ is open in $X$ and $x \in U$ then there is a natural number $n$ such that $st(x, G_n) \neq \emptyset$ and $st(x, G_n) \subseteq U$. If a quasi-developable space is perfect (= closed sets are $G_\delta$ sets) then it is developable [B]. A regular $T_1$ space that is developable is a Moore space. It is shown in [BL] that if $(G_n : n \in \omega)$
is a quasi-development for $X$ and if $x \in U$ where $U$ is open in $X$ then there exists $n$ such that $\emptyset \neq \text{st}(x, G_n) \subseteq U$ and $x$ is an element of only one member of $G_n$.

In this note an example of a quasi-developable 2-manifold that is not developable is given. A different example was independently obtained by Peter Nyikos [Ny2]. Also partial results are proved concerning the metrizability of quasi-developable manifolds.

Let all spaces in this paper be $T_1$-spaces. The following lemma (proved in [RuZ]) is needed to develop techniques used in constructing the example.

**Lemma 1 [RuZ].** Let $\{U_n : n \in \omega\}$ be a nested sequence of open connected subsets of $D' = (-1,1) \cap (0,1)$ such that $\bigcap \{\text{cl}(U_n, D') : n \in \omega\} = \emptyset$ where $\text{cl}(U_n, D')$ denotes the closure of $U_n$ in $D'$ with the relative topology from $\mathbb{R}^2$.

Furthermore let $p_n \in U_n$ for each $n \in \omega$. Then there is a homeomorphism $g$ of $D'$ into $D'$ such that:

1. $D' - g(D')$ is homeomorphic to $J = [0,1)$,
2. $D' - g(D') \subseteq \text{cl}(\{g(p_n) : n \in \omega\}, D')$ and
3. $D' - g(D') \subseteq \text{Int}(\text{cl}(g(U_n), D'), D')$ for each $n \in \omega$.

where $\text{Int}(A, B)$ denotes the interior of $A$ in $B$.

This lemma is a tool in the following definition.

**Definition 1.** Let $M$ be a 2-manifold, $D$ a subspace of $M$ homeomorphic to $D'$, $\{U_n : n \in \omega\}$ a nested sequence of open connected subsets of $D$ with $\bigcap \{\text{cl}(U_n, M) : n \in \omega\} = \emptyset$
and $p_n \in U_n$ for each $n \in \omega$. A Rudin-Zenor extension of $M$ with respect to $D$, $\{U_n: n \in \omega\}$ and $\{p_n: n \in \omega\}$ is a topological space $M'$ described as follows:

Let $g$ be a homeomorphism of $D$ into $D$ as in Lemma 1. Let $g'$ be a homeomorphism of $J$ onto $D - g(D)$ where $J$ is a copy of $[0,1)$ disjoint from $M$. Let $g^*$ be the union of $g$ and $g'$ (thus $g^*$ maps $D \cup J$ onto $D$). Then $M'$ is the unique topological space satisfying:

(i) the underlying set of $M'$ is $M \cup J$,

(ii) $M$ and $J \cup D$ are open in $M'$,

(iii) $M$ keeps its original topology as a subspace of $M'$, and

(iv) the subspace topology on $D \cup J$ is such that $g^*$ is a homeomorphism.

Notice the Rudin-Zenor extension of $M$ adds one copy of $J$ to $M$.

Definition 2. Let $M$ be a 2-manifold and $A$ an index set. Let $V = \{D_\alpha: \alpha \in A\}$ where each $D_\alpha$ is a subspace of $M$ homeomorphic to $D'$. For each $\alpha \in A$ let $U_\alpha = \{U(\alpha, n): n \in \omega\}$ be a decreasing sequence of connected open subsets of $D_\alpha$ such that $\cap \{\text{cl}(U(\alpha, n), M): n \in \omega\} = \emptyset$ and let $U_\alpha = \{U_\alpha: \alpha \in A\}$. For each $\alpha \in A$ and $n \in \omega$, let $p(\alpha, n) \in U(\alpha, n)$ and let $P_\alpha = \{p(\alpha, n): n \in \omega\}$. Let $P = \{P_\alpha: \alpha \in A\}$. Let $J = \{J_\alpha: \alpha \in A\}$ where each $J_\alpha$ is a copy of $[0,1)$, $J_\alpha \cap J_\beta = \emptyset$ if $\alpha \neq \beta$ and each $J_\alpha$ is disjoint from $M$. The free Rudin-Zenor extension of $M$ relative to $(V, U, P, J)$, denoted by $FRZ(M)$, is the unique topological space such that
(i) the underlying set of FRZ(M) is $\bigcup \{ J_\alpha : \alpha \in A \} \cup M$,
(ii) for each $\alpha \in A$, $M \cup J_\alpha$ is an open subspace of FRZ(M), and
(iii) for each $\alpha \in A$ the subspace topology of $M \cup J_\alpha$ is a Rudin-Zenor extension of M.

Notice that FRZ(M) adds $|A|$ many copies of $J$ to $M$ and that FRZ(M) is a $T_1$-space.

Theorem 1. Every free Rudin-Zenor extension is locally $\mathbb{R}^2$. It is Hausdorff (and thus a 2-manifold) if the following property (*) holds:

(*) for each $\alpha, \beta \in A$, $\alpha \neq \beta$, there exists $n \in \omega$ such that

$$\text{cl}(U(\alpha, n), M) \cup \text{cl}(U(\beta, n), M) = 0.$$ 

Proof. FRZ(M) is locally $\mathbb{R}^2$ since, for each $\alpha \in A$, $M \cup J_\alpha$ is a Rudin-Zenor extension of M. The only difficult case for Hausdorffness of FRZ(M) is when $x \in J_\alpha', y \in J_\beta'$ and $\alpha \neq \beta$. Property (*) covers this case.

In order to construct the desired example two topological spaces must be reviewed.

Example 1. (Example 2.17 of Gary Gruenhage's article in [G]). Let B be a Bernstein subset of $\mathbb{R}$ and let

$\{ B_\alpha : \alpha < 2^\omega \}$ be an enumeration of all countable subsets of B such that $\text{cl}(B_\alpha, \mathbb{R})$ is uncountable. For each $\alpha < 2^\omega$ choose
\[ x_\alpha \in \text{cl}(B_\alpha, \mathbb{R}) \setminus (B \cup \{x_\beta: \beta < \alpha\}) \]

and choose points \( x_\alpha(m) \in B_\alpha \) such that the sequence 
\[ (x_\alpha(m): m \in \omega) \]
converges to \( x_\alpha \) in \( \mathbb{R} \). Let \( H = \{x_\alpha: \alpha < 2^\omega\} \) and \( X = B \cup H \). Topologize \( X \) by letting points of \( B \) be isolated and, if \( N(x_\alpha, k) = \{x_\alpha\} \cup \{x_\alpha(m): n \geq k\} \) for each \( k \in \omega \), by letting \( \{N(x_\alpha, k): k \in \omega\} \) be a local base at \( x_\alpha \). Then \( X \) is a locally compact quasi-developable space such that \( H \) is not a \( G_\delta \)-subset of \( X \) (the details of these results are in \([G]\)).

**Example 2.** This example is the Prüfer Manifold \( P(\mathbb{R}) \) ([Ra]) (see example 2.7 of Peter Nyikos' article in \([Nyl]\)). To construct this example collared copies of the real line (i.e. \([0,1) \times \mathbb{R}\)) are attached at each point of the \( x \)-axis to the open upper half plane. Thus the Prüfer manifold as a point set can be visualized as a subset of \( \mathbb{R}^3 \). In fact
\[
P(\mathbb{R}) = \{(x,y,z): x \in \mathbb{R}, y > 0, z = 0\} \cup (\cup \{(x) \times [0,-1) \times \mathbb{R}: x \in \mathbb{R}\}).
\]

Let \( M(x) \) denotes the collared real line that is attached at the point \( x \) on the \( x \)-axis. A Prüfer manifold can be obtained from each subset \( S \) of \( \mathbb{R} \) by attaching an \( M(x) \) to the open upper half plane at each point \( x \) of \( S \). The resulting Prüfer manifold \( P(S) \) is a developable 2-manifold that inherits its topology from \( P(\mathbb{R}) \). Notice that if \( S \) is a countable discrete in itself (i.e. \( S \) contains no limit points) subset of \( \mathbb{R} \) then \( P(S) \) is homeomorphic to \( \mathbb{R}^2 \).
(which is homeomorphic to $D' = (-1,1) \times (0,1)$). Also notice that $P(S)$ as a point set is contained in $\mathbb{R}^3$.

Using these two examples the desired example can be constructed.

**Example 3.** There exists a quasi-developable 2-manifold $Z$ that is not developable.

Consider the set $X = B \cup H$ of Example 1 as a subset of the $x$-axis and let $P(B)$ be the Prüfer 2-manifold constructed over the Bernstein set $B$. Recall that $H = \{x : a < 2^\omega\}$.

For each $a < 2^\omega$, let
\[
D_a = \{(x,y,z) : x \in \mathbb{R}, y > 0, z = 0\} \cup (\cup \{M(x\alpha(n)) : n \in \omega\}).
\]

Since $\{x\alpha(n) : n \in \omega\}$ is discrete in itself as a subset of $\mathbb{R}$, $D_a$ is an open subset of $P(B)$ that is homeomorphic to $D'$. Let $\mathcal{D} = \{D_a : a < 2^\omega\}$.

For each $a < 2^\omega$, let $U(a,n) = A(a,n) \cup B(a,n)$ where
\[
A(a,n) = \{(x,y,z) \in \mathbb{R}^3 : |(x\alpha,0,0) - (x,y,0)| < 1/n, y > 0\}
\]
and
\[
B(a,n) = \cup \{M(x\alpha(m)) : |x\alpha - x\alpha(m)| < 1/n\}.
\]

It follows that $U(a,n)$ is an open connected subset of and that $D_a \cup U(a,n) \supset U(a,n + 1)$ for each $n \in \omega$, and
\[
\cap \{\text{cl}(U(a,n), P(B)) : n \in \omega\} = \emptyset.
\]

Let $U_\alpha = \{U(a,n) : n \in \omega\}$ and $U = \{U_\alpha : a < 2^\omega\}$. Let $p(a,n) = (x\alpha(n),0,0)$ for each $a < 2^\omega$ and $n \in \omega$. Notice
that \( p(\alpha, n) \in U(\alpha, n) \). Let \( P_\alpha = \{ p(\alpha, n): n \in \omega \} \) and \( P = \{ p_\alpha: \alpha < 2^\omega \} \).

Let \( J = \{ J_\alpha: \alpha < 2^\omega \} \) where each \( J_\alpha \) is a copy of \([0, 1)\) disjoint from \( P(B) \) and if \( \alpha \neq \beta \), then \( J_\alpha \cap J_\beta = \emptyset \).

Let \( Z \) be \( \text{FRZ}(P(B)) \) with respect to \((V, U, P, J)\). Notice that \( P(B) \) satisfies property \((*)\). Thus \( \text{FRZ}(P(B)) \) is a 2-manifold.

To see that \( \text{FRZ}(P(B)) \) is not perfect consider the subspace

\[ Y = \bigcup \{ J_\alpha: \alpha < 2^\omega \} \cup \{(x, 0, 0): x \in B\}. \]

Notice that \( B' = \{(x, 0, 0) \in \text{FRZ}(P(B)): x \in B\} \) is an open subset of \( Y \). Hence if \( \text{FRZ}(P(B)) \) was perfect, then \( B' \) would be an \( F_\sigma \)-set in \( Y \). Assume \( B' = \bigcup \{ F'_n: n \in \omega \} \) where \( F'_n \) is closed in \( Y \). There exists \( n \in \omega \) such that \(|F'_n| > \omega\). Let \( F_n = \{ x \in B: (x, 0, 0) \in F'_n \} \). Then \( F_n \) as a closed subset in the space \( X \) of Example 1 contains a \( B_\alpha \).

In this space \( x_\alpha \) is a limit of \( B_\alpha \) and hence of \( F_n \). Thus, in \( Y \), \( J_\alpha \) is contained in \( \overline{\text{cl}(F'_n, Y)} \). But \( J_\alpha \cap B' = \emptyset \). Thus \( B' \) is not an \( F_\sigma \) and it follows that \( \text{FRZ}(P(B)) \) is not perfect.

The following theorem is used to show that \( \text{FRZ}(P(B)) \) is quasi-developable.

**Theorem 1.** Let \( X \) be a regular, locally quasi-developable, \( T_1 \)-space. The following are equivalent:

(i) \( X \) is quasi-developable,

(ii) \( X \) is weakly submetacompact, and
(iii) $X$ has a $\sigma$-relatively discrete cover by quasi-developable sets.

Proof. (i) $\rightarrow$ (ii) see [BL]. For (ii) $\rightarrow$ (iii) let $O(x)$ be an open quasi-developable subset of $X$ containing $x$ for each $x \in X$. Then $\{O(x) : x \in X\}$ has a $\sigma$-relatively discrete refinement (that is also a cover) by quasi-developable subsets. For (iii) $\rightarrow$ (i) let $X = \bigcup \{U^F(n) : n \in \omega\}$ where $F(n) = \{F(n, \alpha) : \alpha \in I_n\}$ is a relatively discrete collection of quasi-developable (hence weakly submetacompact) subsets of $X$. For each $F(n, \alpha) \in F_n$ there exists an open set $U(n, \alpha)$ such that

$$U(n, \alpha) \cap \{U^F(n) : n \in \omega\} = F(n, \alpha).$$

Fix $n$ and $\alpha$ and for each $x \in F(n, \alpha)$ let $O(x)$ be an open quasi-developable set that contains $x$ such that $O(x) \subset U(n, \alpha)$. Since $\{O(x) \cap F(n, \alpha) : x \in F(n, \alpha)\}$ is an open cover of $F(n, \alpha)$ it has a $\sigma$-relatively discrete refinement $R(n, \alpha) = \{R(n, \alpha, k) : k \in \omega\}$ that covers $F(n, \alpha)$. Fix $k$. For each $R \in R(n, \alpha, k)$ let $V(R)$ be an open set in $X$ such that

$$\{V(R) \cap F(n, \alpha) : R \in R(n, \alpha, k)\}$$

witnesses that $R(n, \alpha, k)$ is a relatively discrete collection. If $R \in R(n, \alpha, k)$ let $x(R) \in F(n, \alpha)$ such that $R$ refines $O(x(R))$. Let $\{G(n, \alpha, k, R, m) : m \in \omega\}$ be a quasi-development for $O(x(R)) \cap V(R) \cap U(n, \alpha)$. Let

$$H(n, k, m) = \{G \in \{G(n, \alpha, k, R, m) : F(n, \alpha) \in F_n, R \in R(n, \alpha, k)\}$$

Then $H = \{H(n, k, m) : n \in \omega, k \in \omega, m \in \omega\}$ is a
quasi-development for X. To see this let \( x \in U \) where \( U \) is open in \( X \). There exists \( n \) and \( \alpha \) such that \( x \in F(n,\alpha) \) and there exists \( k \in \omega \) and \( R \in R(n,\alpha,k) \) such that \( x \in R \).

Then there exists \( m \) such that

\[
\text{st}(x, G(n,\alpha,k,R,m)) \subset U \cap O(x(R)) \cap V(R) \cap U(n,\alpha).
\]

Hence \( \text{st}(x, H(n,k,m)) \subset U \).

Notice that the underlying set in \( FRZ(P(B)) \) is \( P(B) \cup (\bigcup \{ J_\alpha : \alpha < 2^\omega \}) \). Since \( P(B) \) as a subspace is developable it has a \( \sigma \)-relatively discrete cover and since \( (J_\alpha : \alpha < 2^\omega) \) is a pairwise disjoint collection it is \( \sigma \)-relatively discrete. Since \( FRZ(P(B)) \) is a manifold it is locally quasi-developable. Hence, by the preceding theorem, \( FRZ(P(B)) \) is quasi-developable.

The same argument as Peter Nyikos gives in [Ny1] shows that \( FRZ(P(B)) \) is not normal.

The following question remains open:

**Question 1.** Is every hereditarily normal quasi-developable manifold paracompact?

A partial affirmative answer is given if \( 2^{\omega_1} > 2^\omega \).

**Theorem 2.** Assume \( 2^{\omega_1} > 2^\omega \). Every hereditarily normal quasi-developable manifold is paracompact.

Note that an actually stronger result was announced without proof by one of the authors (see the remark after Theorem 2.5 together with Lemma 2.1 in [Ba]).
According to that result "quasi-developable manifold" can be weakened to "connected, locally c.c.c., hereditarily weakly submeta-Lindelöf space" in Theorem 2 (weakly submeta-Lindelöf = weakly $\delta^\beta$-refinable). Since the proof of the more general result has not appeared in print we feel justified in giving a proof of Theorem 2 here.

Proof of Theorem 2. First recall a result of Taylor [Ta] showing each first-countable hereditarily normal space has the following property under $2^{\omega_1} > 2^\omega$:

(*) if $C$ is a cub subset of $\omega_1$ and $\{x_\alpha: \alpha \in C\}$ is a weakly $\sigma$-discrete set of distinct points then there is a stationary subset $S \subseteq C$ such that $\{x_\alpha: \alpha \in S\}$ has an expansion by pairwise disjoint open sets.

Now suppose indirectly that there is a non-paracompact, hereditarily normal, quasi-developable manifold $X$. Then $X$ has a connected open submanifold $Y$ of weight $\omega_1$. Let $\{U_\alpha: \alpha \in \omega_1\}$ be an open cover of $Y$ by separable open subsets. Since $Y$ is connected we can choose, for each $\alpha \in \omega_1$, a point

$$y_\alpha \in \text{cl}(\bigcup\{U_\beta: \beta < \alpha\}) \setminus \bigcup\{U_\beta: \beta < \alpha\}.$$ 

Let $C$ be a cub subset of $\omega_1$ such that $L = \{y_\alpha: \alpha \in C\}$ consists of distinct points. Note that $L$ is locally countable and, thus, a $\sigma$-scattered space which is hereditarily weak submetacompact and, hence, weakly $\sigma$-discrete ([Ny2], Corollary 3.5). By (*) there is a stationary set $S \subseteq \omega_1$ such that $\{y_\alpha: \alpha \in S\}$ has a pairwise disjoint expansion $\{B_\alpha: \alpha \in S\}$ by open sets. Since
for each $a \in S$ there is an $f(a) < a$ such that $B_a \cap U_{f(a)} \neq \emptyset$. By the pressing down lemma there is a $\beta \in \omega_1$ such that $f(a) = \beta$ for uncountably many $a \in S$. Therefore uncountably many of the $B_\alpha$'s intersect $U_\beta$ violating the separability of $U_\beta$.

References


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