$\epsilon$-MAPPINGS ONTO A TREE AND THE FIXED POINT PROPERTY

by

M. M. Marsh
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In 1979 David Bellamy [1] showed that there exist tree-like continua which admit fixed point free mappings. There has been interest since that time in determining conditions under which a tree-like continuum will have the fixed point property. A few results of this nature can be found in [2], [3], [4], [7], [8], and [9]. However, it is still unknown if a simple triod-like continuum must have the fixed point property. This paper establishes several fixed point related theorems for T-like continua, where T is a fixed tree. Corollary 3 gives a necessary condition for a T-like continuum to admit a fixed point free mapping, and Theorem 2 generalizes the fixed point theorem in [7].

A continuum is a nondegenerate compact connected metric space. A continuous function will be referred to as a map or mapping. A continuum X has the fixed point property provided that whenever f is a mapping of X into itself, there is a point x in X such that f(x) = x. A tree is a finite connected, simply connected graph. If \epsilon is a positive number, the mapping f: X \to Y is an \epsilon-mapping if diam(f^{-1}(y)) < \epsilon for each y \in Y. If H is a family of continua, we say that the continuum X is H-like provided that, for each positive number \epsilon, there is an
\(\varepsilon\)-mapping of \(X\) onto a member of \(H\). For example, if \(H\) is the family of all trees, we simply say that \(X\) is tree-like; or if \(H\) is a set whose only member is the continuum \(T\), we say that \(X\) is \(T\)-like.

Let \(T\) be a tree. The point \(v \in T\) is a branchpoint (an endpoint) of \(T\) if \(T - \{v\}\) has at least three components (only one component). If \(v\) is either a branchpoint or an endpoint of \(T\), we say that \(v\) is a vertex of \(T\). If \(v\) and \(w\) are points of \(T\), let \([v, w]\) denote the arc in \(T\) with endpoints \(v\) and \(w\), and let \(T(v, w]\) denote the component of \(T - \{v\}\) that contains \(w\).

Lemma. Let \(F\) be a function from the vertex set of the tree \(T\) into the set of all subsets of \(T\). If for each vertex \(v\) of \(T\), \(F(v)\) is a subset of the closure of some component of \(T - \{v\}\), then there exist neighboring (adjacent) vertices \(v\) and \(w\) in \(T\) such that \(F(v) \subseteq T(v, w]\) and \(F(w) \subseteq T(w, v]\).

Proof. Let \(v_1\) be any branchpoint of \(T\) and let \(C_1\) be the component of \(T - \{v_1\}\) such that \(F(v_1)\) is a subset of \(C_1\). Let \(v_2\) be the vertex of \(C_1\) that is adjacent to \(v_1\). So, \(C_1 = T(v_1, v_2]\). If \(F(v_2) \subseteq T(v_2, v_1]\), then \(v_1\) and \(v_2\) have the desired properties. Otherwise, \(v_2\) must be a branchpoint of \(T\) and there is a component \(C_2\) of \(T - \{v_2\}\) such that \(C_2 \neq T(v_2, v_1]\) and \(F(v_2) \subseteq C_2\). Now, \(C_2 \subseteq C_1\) and \(C_2\) contains fewer branchpoints than \(C_1\). Since \(C_1\) has finitely many branchpoints, a repetition of the process above must yield adjacent vertices with the desired properties.
We introduce the following terminology. Given a sequence \( \{F_n\}_{n=1}^{\infty} \), to say that

\( \{F_n\}_{n=1}^{\infty} \) \textit{frequently} has some property means that for each positive integer \( N \), there is an integer \( n \geq N \) such that \( F_n \) has the property,

and to say that

\( \{F_n\}_{n=1}^{\infty} \) \textit{eventually} has some property means that there is a positive integer \( N \) such that if \( n \geq N \), then \( F_n \) has the property.

We are now ready for our main theorems.

**Theorem 1.** Suppose that \( T \) is a tree, \( X \) is \( T \)-like, and for each \( n \geq 1 \), \( g_n: X \rightarrow T \) is a \( \delta_n \)-mapping onto \( T \), where \( \{\delta_n\}_{n=1}^{\infty} \) converges to zero. If \( f: X \rightarrow X \) is a mapping, \( \{n_i\}_{i=1}^{\infty} \) is an increasing sequence of positive integers, and there are adjacent vertices \( v \) and \( w \) of \( T \) such that

\( \{g_{n_i}f_{n_i}^{-1}(v)\}_{i=1}^{\infty} \) is eventually a subset of \( T(v,w) \) and

\( \{g_{n_i}f_{n_i}^{-1}(w)\}_{i=1}^{\infty} \) is eventually a subset of \( T(w,v) \), then \( f \) has a fixed point.

**Proof.** Suppose that \( f \) is fixed point free. Let \( d \) denote the metric on \( X \). Assume that each edge of \( T \) has length one and let \( p \) denote the "arc length" metric on \( T \). Let \( \epsilon \) be a positive number such that \( d(x,f(x)) \geq \epsilon \) for each \( x \in X \).

Fix \( n \) large enough so that \( g_n \) is an \( \frac{\epsilon}{2} \)-mapping,

\( g_n f g_n^{-1}(v) \subseteq T(v,w) \), and \( g_n f g_n^{-1}(w) \subseteq T(w,v) \). Since \( g_n \) is
an $\varepsilon/2$-mapping, it follows that $t \not\in g_n f g_n^{-1}(t)$ for any $t \in T$. So, we have that $g_n f g_n^{-1}(v) \subseteq T(v,w)$ and $g_n f g_n^{-1}(w) \subseteq T(w,v)$.

Let $0 < \delta < 1$ such that if $d(x,y) \geq \varepsilon$, then $p(g_n(x), g_n(y)) > \delta$. That such a $\delta$ exists is easily seen (argument by contradiction).

Let $V$ be an open set in $X$ such that $g_n^{-1}(v) \subset V$, $\text{diam} V < \varepsilon$, and if $x \in V$, then $g_n f(x) \in T(v,w)$ and $p(g_n(x), v) < \frac{\delta}{2}$. Similarly, let $W$ be an open set in $X$ such that $g_n^{-1}(w) \subset W$, $\text{diam} W < \varepsilon$, and if $x \in W$, then $g_n f(x) \in T(w,v)$ and $p(g_n(x), w) < \frac{\delta}{2}$.

Pick any point $q$ in $g_n^{-1}(w)$ and let $L$ be the component of $X - V$ that contains $q$. Now, $L$ must intersect the boundary of $V$ at some point $y$. We point out that $g_n(L) \subset T(v,w)$. For if not, there is a point $x \in L$ such that $g_n(x) \in T(w,v) - (v,w)$. Also, $q \in L$ and $g_n(q) = w$. Since $L$ is connected and $g_n$ is continuous, it follows that there is a point of $L$ that is also in $g_n^{-1}(v) \subset V$, a contradiction.

Let $K$ be the component of $L - W$ that contains $y$. Let $z$ be a point of the boundary of $W$ that is also in $K$. As above, $g_n(K) \subset T(w,v)$. For if not, there is a point $x \in K$ such that $g_n(x) \in T(v,w) - [v,w)$. Since $y \in \overline{V}$, $g_n(y) \in T(w,v)$. Now, $y$ is also in $K$; hence, there is a point of $K$ that is also in $g_n^{-1}(w) \subset W$, a contradiction.

Since $K \subset L$, we get that $g_n(K) \subset (v,w)$. Let $R = \{x \in K | g_n(x) \text{ separates } g_n f(x) \text{ from } v \text{ in } T\}$
and

\[ S = \{ x \in K | g_n(x) \text{ separates } g_n(f(x)) \text{ from } w \text{ in } T \}. \]

Clearly, \( R \cup S = K \), and \( R \) and \( S \) are disjoint open sets in \( K \). We will show that \( y \in R \) and \( z \in S \).

Suppose that \( y \notin R \). Then \( y \in S \) and \( g_n(y) \) must separate \( g_n(f(y)) \) from \( w \) in \( T \). Since \( y \in \overline{V} \), \( p(g_n(y),v) \leq \frac{\delta}{2} \)
and \( g_n(f(y)) \in \overline{T(v,w)} \). Hence, we must have that \( g_n(f(y)) \in [v,w] \) and that \( p(g_n(f(y)),g_n(y)) \leq \frac{\delta}{2} < \delta \). But, by choice
of \( \delta \), \( d(y,f(y)) \geq \varepsilon \) implies that \( p(g_n(y),g_n(f(y)) \geq \delta \), a contradiction.

A symmetric argument gives us that \( z \in S \). But then
\( K \) is not connected, which is a contradiction.

Since an arc is a tree with exactly two vertices, namely its endpoints, we get Hamilton's [5] fixed point theorem as an immediate corollary.

**Corollary 1.** If \( X \) is an arc-like continuum, then \( X \) has the fixed point property.

**Corollary 2.** Suppose that \( T \) is a simple \( k \)-od with branchpoint \( v \), \( X \) is \( T \)-like, and for each \( n \geq 1 \), \( g_n : X \to T \)
is a \( \delta_n \)-mapping onto \( T \), where \( \{ \delta_n \}_{n=1}^{\infty} \) converges to zero.
If \( f : X \to X \) is a fixed point free mapping, then
\( \{ g_n f g_n^{-1}(v) \}_{n=1}^{\infty} \) eventually intersects two components of
\( T - \{ v \} \).

**Proof.** Suppose that \( \{ g_n f g_n^{-1}(v) \}_{n=1}^{\infty} \) does not eventually intersect two components of \( T - \{ v \} \). Then there is
a component \( L \) of \( T - \{ v \} \) such that \( \{ g_n f g_n^{-1}(v) \}_{n=1}^{\infty} \) is
frequently a subset of $L$. Let $e$ be the endpoint of $T$ that belongs to $L$. Then $v$ and $e$ are adjacent vertices of $T$. Also, $(g_n f g_n^{-1}(e))_{n=1}^\infty$ is a subset of $\overline{T(e,v]}$ for all $n \geq 1$ since $T(e,v] = T - \{e\} = T$. It follows from Theorem 1 that $f$ has a fixed point, which is a contradiction.

**Corollary 3.** Suppose that $T$ is a tree, $X$ is $T$-like, and for each $n \geq 1$, $g_n : X \to T$ is a $\delta_n$-mapping onto $T$, where $\{\delta_n\}_{n=1}^\infty$ converges to zero. If $f : X \to X$ is a fixed point free mapping, then there is a branchpoint $v$ of $T$ such that $(g_n f g_n^{-1}(v))_{n=1}^\infty$ frequently intersects two components of $T - \{v\}$.

**Proof.** By way of contradiction, we assume that for each branchpoint $v$ of $T$, there is a positive integer $N_v$ such that if $n \geq N_v$, then $g_n f g_n^{-1}(v)$ is a subset of the closure of some component of $T - \{v\}$.

Let $N = \max\{N_v \mid v$ is a branchpoint of $T\}$ and fix $n \geq N$. We recall that if $e$ is an endpoint of $T$ and $v$ is the vertex of $T$ adjacent to $e$, then $g_n f g_n^{-1}(e) \subseteq \overline{T(e,v]}$. Hence, by the lemma, there exist adjacent vertices $v$ and $w$ in $T$ such that $g_n f g_n^{-1}(v) \subseteq \overline{T(v,w]}$ and $g_n f g_n^{-1}(w) \subseteq \overline{T(w,v]}$. So, if $n \geq N$, we may associate with $n$ a pair of adjacent vertices in $T$ that have the properties above. Since there are only finitely many pairs of adjacent vertices in $T$, it follows that there is an increasing sequence $\{n_i\}_{i=1}^\infty$, each term of which is associated with the same pair of adjacent vertices. By Theorem 1, $f$ has a fixed point, which is a contradiction.
Our next theorem generalizes, in the case of finite fans, the fixed point result in [7].

**Theorem 2.** Let $T$ be a tree, and for each branchpoint $v$ of $T$, let $(L_i(v))_{i=1}^{k_v}$ be a labeling of the components of $T - \{v\}$. If $X = \lim\{T, g_{n+1}^n\}$, where for each $n \geq 1$ and each branchpoint $v$ of $T$, $g_{n+1}^n(L_i(v)) = L_i(v)$ for $2 \leq i \leq k_v$, then $X$ has the fixed point property.

**Proof.** Let $d$ denote the metric on $X$ and, for each $n \geq 1$, let $g_n$ be the projection mapping of $X$ onto $T$. Now, $X$ is $T$-like and for $\varepsilon > 0$, $n$ can be chosen so that $g_n$ is an $\varepsilon$-mapping (see [6]).

By way of contradiction, we assume that $f$ is a fixed point free mapping on $X$ and that $\varepsilon$ is a positive number such that $d(x, f(x)) \geq \varepsilon$ for each $x \in X$.

Let $v$ be any branchpoint of $T$. We notice that $g_{n+1}^n(v) = v$ for each $n \geq 1$. So, let $p_v$ be the point of $X$ such that $g_n(p_v) = v$ for each $n \geq 1$. Also, let $M_v = \bigcup_{i=2}^{k_v} L_i(v)$. We further observe that

\[ (*) \text{ if } x \in X \text{ and there is an integer } N \text{ such that } g_N(x) \text{ is not in } M_v, \text{ then for } n \geq N, \ g_n(x) \notin M_v. \]

Suppose that $(*)$ is not the case. Then there is a point $x \in X$ and positive integers $N$ and $n$ with $n \geq N$ such that $g_N(x) \notin M_v$ but $g_n(x) \in M_v$. However, this implies that $g_N(x) = g_{n+1}^n(x) \in M_v$, which is a contradiction.
Hence, by (*) and the fact that $g_{n+1}^{n+1}(M_v) \subseteq M_v$ for each $n \geq 1$, we may choose a positive integer $m$ such that $g_m$ is an $\varepsilon$-mapping and so that either

i) $g_n(f(p_v)) \in L_1(v)$ for $n \geq m$ or

ii) $g_n(f(p_v)) \in M_v$ for $n \geq m$.

Note that $g_n(f(p_v)) \neq v$, for $n \geq m$, since $g_m$ is an $\varepsilon$-mapping and $v = g_n(p_v)$. Since $g_{n+1}^{n+1}(L_1(v)) = L_1(v)$ for $n \geq 1$ and $2 \leq i \leq k_v$, it follows that if $g_n(f(p_v)) \in M_v$ for $n \geq m$, then there is an integer $2 \leq j \leq k_v$ such that $g_n(f(p_v)) \in L_j(v)$ for $n \geq m$. So, in fact, we have that there is an integer $1 \leq i \leq k_v$ such that $g_n(f(p_v)) \in L_i(v)$ for $n \geq m$.

Let $\delta$ be a positive number such that if $x \in X$ and $d(x, p_v) < \delta$, then $g_m f(x) \in L_1(v)$. Let $n \geq m$ and large enough so that $g_n$ is a $\delta$-mapping. Since $p_v \in g_{n-1}^{-1}(v)$ and $\text{diam}(g_{n-1}(v)) < \delta$, it follows that if $x \in g_{n-1}^{-1}(v)$, then $d(x, p_v) < \delta$ and $g_m f(x) \in L_1(v)$. Thus, $g_m f g_{n-1}^{-1}(v) \subseteq L_1(v)$. Now, if $i = 1$, then by (*), $g_m f g_{n-1}^{-1}(v) \subseteq L_1(v)$. If $i \neq 1$, we get that $g_m f g_{n-1}^{-1}(v) \subseteq L_1(v) \cup L_i(v)$.

We have shown that for each branchpoint $v$ of $T$, there is a positive integer $m_v$ and an integer $1 \leq i_v \leq k_v$ such that for $n \geq m_v$,

1) $g_n(f(p_v)) \in L_{i_v}(v)$, and

2) $g_n f g_{n-1}^{-1}(v) \subseteq L_1(v) \cup L_{i_v}(v)$.  


Let $N = \max(m_v \mid v$ is a branchpoint of $T)$. For $n \geq N$, and $v$ a branchpoint of $T$, let

$$F_n(v) = \begin{cases} g_n f_{g_n}^{-1}(v) & \text{if } g_n f_{g_n}^{-1}(v) \text{ intersects only one of } \\ g_n f_{g_n}^{-1}(v) \cap L_1(v) & \text{otherwise.} \end{cases}$$

For $n \geq N$ and $e$ an endpoint of $T$, let $F_n(e) = g_n f_{g_n}^{-1}(e)$.

By our lemma, for each $n \geq N$, there are adjacent vertices $v$ and $w$ of $T$ such that $F_n(v) \subseteq T(v,w)$ and $F_n(w) \subseteq T(w,v)$. By the finiteness of the set of all pairs of adjacent vertices in $T$, we can pick an increasing number sequence $\{n_i\}_{i=1}^{\infty}$ and a pair of adjacent vertices $v$ and $w$ such that for each $i \geq 1$, $F_{n_i}(v) \subseteq T(v,w)$ and $F_{n_i}(w) \subseteq T(w,v)$. Let $\leq$ be a partial order on $T$ that is consistent with the metric on $T$ and such that $v$ is the least element of $T(v,w)$ and $w$ is the maximum element of $T(w,v)$.

The remainder of the proof involves three cases.

**Case 1.** $\{g_{n_i} f_{g_{n_i}}^{-1}(v)\}_{i=1}^{\infty}$ eventually intersects only one of $L_1(v)$ and $L_{1,v}$, and $\{g_{n_i} f_{g_{n_i}}^{-1}(w)\}_{i=1}^{\infty}$ eventually intersects only one of $L_1(w)$ and $L_{1,w}$.

In this case, by definition, $F_{n_i}(v) = g_{n_i} f_{g_{n_i}}^{-1}(v)$ and $F_{n_i}(w) = g_{n_i} f_{g_{n_i}}^{-1}(w)$ for all $i$ beyond some integer. It follows from Theorem 1 that $f$ has a fixed point, which is a contradiction.
Case 2. $\{g_{n_i} f g_{n_i}^{-1}(v)\}_{i=1}^{\infty}$ frequently intersects both of $L_i(v)$ and $L_{i_v}(v)$ and $\{g_{n_i} f g_{n_i}^{-1}(w)\}_{i=1}^{\infty}$ frequently intersects both of $L_i(w)$ and $L_{i_w}(w)$.

We observe that if $i_v \neq 1$ and $g_r f g_k^{-1}(v)$ intersects $L_i(v)$ for any integer $r$, then $g_k f g_k^{-1}(v)$ intersects $L_i(v)$ for each integer $k \leq r$. To see this, let $k \leq r$ and first notice that $g_r^{-1}(v) \subseteq g_k^{-1}(v)$ since $v$ is fixed by all bonding mappings. Thus, $g_r f g_k^{-1}(v) \subseteq g_k f g_k^{-1}(v)$. So, there is a point $x$ in $L_i(v) \cap g_k f g_k^{-1}(v)$. Since $i_v \neq 1$, $g_k^{-1}(x) \in L_i(v)$. Hence, $g_k^{-1}(g_r f g_k^{-1}(v)) = g_k f g_k^{-1}(v)$ intersects $L_i(v)$.

By our assumption in this case, $i_v \neq 1$ and $i_w \neq 1$. Hence, since $\{g_{n_i} f g_{n_i}^{-1}(u)\}_{i=1}^{\infty}$ frequently intersects $L_i(u)$ for $u \in \{v, w\}$, it follows from our observation in the preceding paragraph that $\{g_{n_i} f g_{n_i}^{-1}(u)\}_{i=1}^{\infty}$ intersects $L_i(u)$ for all $n \geq 1$. So, by definition, $F_n(u) \subseteq L_i(u)$ for all $n \geq 1$ and $u \in \{v, w\}$. It follows that $L_i(v) = T(v, w]$ and $L_i(w) = T(w, v]$. Hence, for each $n \geq 1$, $g_{n+1}^{n+1}(T(v, w]) = T(v, w]$ and $g_{n+1}^{n+1}(T(w, v]) = T(w, v]$. It follows that for $n \geq 1$, $g_{n+1}([v, w]) = [v, w]$.

Let $C = \lim \{[v, w], g_{n+1}^{n+1}|[v, w]\}$. Now, $C$ is an arc-like continuum containing the points $p_v$ and $p_w$. Recall that for each $n \geq n_v$, $g_n f(p_v) \in L_i(v) = T(v, w]$ and for $n \geq n_w$, $g_n f(p_w) \in L_i(w) = T(w, v]$. Let $n$ be large enough
so that \( n > \max\{m_v, m_w\} \) and \( g_n \) is an \( \varepsilon \)-mapping. Let
\[
R = \{ x \in C \mid g_n(x) < g_n f(x) \}
\]
and
\[
S = \{ x \in C \mid g_n(x) > g_n f(x) \}.
\]
Clearly, \( R \cup S = C \), \( R \) and \( S \) are open disjoint sets in \( C \), \( p_v \in R \), and \( p_w \in S \). But then \( C \) is not connected, which is a contradiction.

**Case 3.** \( \{ g_n^{-1}(v) \}_{i=1}^{\infty} \) eventually intersects only one of \( L_1(v) \) and \( L_{i_v}(v) \), and \( \{ g_n^{-1}(w) \}_{i=1}^{\infty} \) frequently intersects both of \( L_1(w) \) and \( L_{i_w}(w) \).

As in Case 2, it follows that \( i_w \neq 1 \), \( L_{i_w}(w) = T(w,v] \), and \( F_n(w) \subseteq T(w,v] \) for all \( n \geq 1 \).

Now, if \( i_v \neq 1 \) and \( \{ g_n^{-1}(v) \}_{i=1}^{\infty} \) is frequently a subset of \( L_{i_v}(v) \), then the argument beginning with the second paragraph in Case 2 applies and we are done. So, we may assume that \( \{ g_n^{-1}(v) \}_{i=1}^{\infty} \) is eventually a subset of \( L_1(v) \). Thus, for all \( i \) beyond some integer, \( F_n(v) = g_n^{-1}(v) \), and it follows that \( L_1(v) = T(v,w] \). We may choose an integer \( n \) large enough so that \( n \geq m_w \), \( g_n^{-1}(v) \subseteq \overline{T(v,w]} \), \( g_n^{-1}(w) \cap T(w,v] \neq \emptyset \), and \( g_n \) is an \( \varepsilon \)-mapping. Let \( \delta \) be a positive number such that
\[
d(x,y) \geq \varepsilon \text{ in } X \text{ implies that } p(g_n(x), g_n(y)) \geq \delta \text{ in } T.
\]
Let \( V \) be an open set in \( X \) such that \( g_n^{-1}(v) \subseteq V \), \( \text{diam} V < \varepsilon \), and if \( x \in V \), then \( g_n f(x) \in T(v,w] \) and
Let \( M = \lim_{i \to \infty} \{ \overline{T(w,v)} : g_i^{i+1} \} \), and let \( C \) be the component of \( M - V \) that contains \( p_w \). Recall that \( g_n f(p_w) \in L_i \{ w \} = T(w,v) \) since \( n \geq m \). Now, \( C \) must intersect the boundary of \( V \) at some point \( y \). We point out that \( g_n(C) \subseteq \overline{T(v,w)} \). For if not, there is a point \( x \in C \) such that \( g_n(x) \in T(w,v) - [v,w] \). Also, \( p_w \in C \) and \( g_n(p_w) = w \).

Since \( C \) is connected and \( g_n \) is continuous, it follows that there is a point of \( C \) that is also in \( g_n^{-1}(v) \subseteq V \), a contradiction.

Furthermore, \( g_n(C) \subseteq \overline{T(w,v)} \) simply because \( C \subseteq M \).

It follows that \( g_n(C) \subseteq [v,w] \). Let
\[
R = \{ x \in C \mid g_n(x) < g_n f(x) \}
\]
and
\[
S = \{ x \in C \mid g_n(x) > g_n f(x) \}.
\]
Clearly, \( R \cup S = C \), and \( R \) and \( S \) are disjoint open sets in \( C \). We will show that \( y \in R \) and \( p_w \in S \).

Now, \( p_w \in S \) since \( g_n(p_w) = w \) and \( g_n f(p_w) \in T(w,v) \).

Suppose \( y \notin R \). Then \( y \in S \) and \( g_n(y) > g_n f(y) \).

Since \( y \in V \), \( p(g_n(y),v) \leq \frac{\delta}{2} \), and \( g_n f(y) \in \overline{T(v,w)} \). Hence, we must have that \( g_n f(y) \in [v,w] \) and that \( p(g_n f(y),g_n(y)) \leq \frac{\delta}{2} < \delta \). But by choice of \( \delta \), \( d(y,f(y)) \geq \epsilon \) implies that \( p(g_n(y),g_n f(y)) \geq \delta \), a contradiction.

But now we have that \( C \) is not connected, which is a contradiction.
References


California State University
Sacramento, California 95819-6051