CONFLUENT MAPPINGS ON [0, 1] AND INVERSE LIMITS

by

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ABSTRACT. Let $I = [0,1]$. In this paper, confluent mappings $f : I \rightarrow I$ are characterized. The degree, $\text{deg}(f)$ of such a mapping is defined as $(\text{number of components of } f^{-1}(0)) + (\text{number of components of } f^{-1}(1)) - 1$. This definition agrees with the usual definition of $\text{deg}(f)$ in the case where $f$ is open. It is shown that, if $f_i : I \rightarrow I$ is confluent, $g_i : I \rightarrow I$ is open and $\text{deg}(f_i) = \text{deg}(g_i)$ for $i = 1, 2, \ldots$, then $\lim\{I, f_i\}$ is homeomorphic with $\lim\{I, g_i\}$.

The simplest indecomposable continua are those which can be constructed as inverse limits of $I = [0,1]$, with open bonding maps. In this paper we show that inverse limits on $I$ with confluent bonding maps are homeomorphic to inverse limits on $I$ with open bonding maps, and we identify the particular inverse limit of the latter type that a given inverse limit with confluent bonding maps is homeomorphic to. This weakening of the condition on the bonding map is useful: it makes it much easier to construct mappings onto these continua (see [2] for example). To prove this result, we first obtain a characterization of confluent mappings from $I$ onto $I$ which is similar to the characterization of open mappings in [8]. From this characterization it follows that such confluent mappings are uniformly approximated by open mappings. Given an inverse limit, $X$, on $I$ with confluent bonding maps we may apply a theorem of Morton Brown [1] to obtain a homeomorphism form $X$ onto an inverse limit on $I$ with open bonding maps. The degrees of the corresponding confluent mappings and open mappings are the same, and thus the inverse limit with open bonding maps
A classification of inverse limits on $I$ with a fixed open bonding map was obtained independently by W. Debski [3] and W.T. Watkins [8]. A classification in the more general case, allowing different bonding maps, was obtained by Debski. It follows as a corollary to the result discussed above that the inverse limits on $I$ with confluent bonding maps are classified in the same way.

All spaces considered in this paper are metric. A continuum is a compact connected metric space and a mapping is a continuous function. A mapping $f$ from the space $X$ onto the space $Y$ is confluent provided that, for each subcontinuum $K$ of $Y$, each component of $f^{-1}(K)$ is mapped by $f$ onto $K$. We will adopt the following notational conventions: if $a = b$, we define $[a, b] = \{a\}$; if $H_1$ and $H_2$ are mutually exclusive sets of real numbers then we say that $H_2$ is to the right of $H_1$ and write $H_1 < H_2$ if it is true that $s < t$ for all $s \in H_1$ and $t \in H_2$; if $H_1, H_2$ and $H_3$ are mutually exclusive sets of real numbers then we say that $H_2$ is between $H_1$ and $H_3$ if either $H_1 < H_2 < H_3$ or $H_3 < H_2 < H_1$.

Our first lemma follows easily from the Intermediate Value Theorem and the definition of confluent mappings.

**Lemma 1.** Suppose that $f : I \to I$ is confluent, $a$ and $b$ are in $I$, $a < b$ and $f(a) = f(b)$.

1. If there is a number $x$ between $a$ and $b$ such that $f(x) > f(a)$ then there is a number $c$ between $a$ and $b$ such that $f(c) = 1$.

2. If there is a number $x$ between $a$ and $b$ such that $f(x) < f(a)$ then there is a number $c$ between $a$ and $b$ such that $f(c) = 0$.

**Lemma 2.** Suppose that $f : I \to I$ is confluent, $a$ and $b$ are in $I$, $a < b$ and $f(a) = f(b)$.

1. If $f(x) > f(a)$ for all $x$ in $(a, b)$ then there is just one component of $f^{-1}(1)$ which is between $a$ and $b$.

2. If $f(x) < f(b)$ for all $x$ in $(a, b)$ then there is just one component of $f^{-1}(0)$ which is between $a$ and $b$.

**Proof.** This is a straight forward application of Lemma 1.
From Lemma 2 and the uniform continuity of continuous functions on $I$ we obtain the following lemma.

**Lemma 3.** Suppose that $f : I \rightarrow I$ is confluent. Then there exists $\epsilon > 0$ such that if $a$ and $b$ are in $I$ and $f(a) = f(b) = 0$ or $f(a) = f(b) = 1$ and $f$ is not constant on $[a, b]$ then $|b - a| > \epsilon$.

**Corollary** Suppose that $f : I \rightarrow I$ is confluent. Then $f^{-1}(0)$ and $f^{-1}(1)$ each have only finitely many components.

**Lemma 4.** If $f : I \rightarrow I$ is confluent then either $f(0) = 0$ or $f(0) = 1$ and either $f(1) = 0$ or $f(1) = 1$.

**Proof.** We will prove the first conclusion only. Suppose $f(0) \neq 0$ and $f(0) \neq 1$. Let

$$x = \text{g.l.b.}(f^{-1}(0) \cup f^{-1}(1))$$

Then either $f(x) = 0$ or $f(x) = 1$. In either case $x > 0$. Suppose that $f(x) = 0$. Let $K = [f(0)/2, 1]$. Let $L$ be the component of $f^{-1}(K)$ which contains 0. Since $f$ is confluent, there is a point $c$ in $L$ such that $f(c) = 1$. Since $f(0)/2 > 0 = f(x)$ we have that $L \subset [0, x]$, and thus $c < x$. This is inconsistent with the definition of $x$.

**Lemma 5.** Suppose that $f : I \rightarrow I$ is confluent, $f^{-1}(0)$ has $n$ components, $L_1 < L_2 < L_3 < \ldots < L_n$, and $f^{-1}(1)$ has $m$ components, $H_1 < H_2 < H_3 < \ldots < H_m$. Then the components of $f^{-1}(0)$ and $f^{-1}(1)$ alternate, and

1. if $0 \in L_1$ and $1 \in H_m$ then $m = n$,
2. if $0 \in H_1$ and $1 \in L_n$ then $m = n$,
3. if $0 \in L_1$ and $1 \in L_n$ then $n = m + 1$, and
4. if $0 \in H_1$ and $1 \in H_m$ then $m = n + 1$.

**Proof:** That the components of $f^{-1}(0)$ and $f^{-1}(1)$ alternate follows from Lemma 2. Suppose 0 is in $L_1$. Thus no $H_i$ is to the left of $L_1$. Since $H_1, H_2, \ldots, H_m$ alternates with $L_1, L_2, L_3, \ldots, L_n$, there are $n - 1$ $H_i$s to the left of $L_n$. There can be no more than one $H_i$ to the right of $L_n$. If 1 is in $L_n$, there are no $H_i$s to the right of $L_n$. Thus if 1 is in $L_n$, then $m = n - 1$ and we have (3). If 1 is not in $L_n$, then 1 is in $H_m$ by Lemma
4. Therefore \( m = n \) and we have (1). Similar reasoning yields (2) and (4).

**Definition.** Suppose that \( f : I \rightarrow I \) is confluent, \( m \) is the number of components of \( f^{-1}(0) \), and \( n \) is the number of components of \( f^{-1}(1) \). The degree of \( f \), denoted \( \deg(f) \), is the integer \( m + n - 1 \). Note that if \( f \) is an open mapping, then \( \deg(f) \) as defined in [3, p. 204] agrees with this definition.

**Theorem 1.** A mapping \( f : I \rightarrow I \) is confluent if and only if there is a positive integer \( n \) and sequences \( a_0, a_1, \ldots, a_n \) and \( b_0, b_1, \ldots, b_n \) with

\[
0 = a_0 \leq b_0 < a_1 \leq b_1 < \ldots < a_{n-1} \leq b_{n-1} < a_n \leq b_n = 1,
\]

such that, for \( i = 1, 2, \ldots, n \), \( f \) restricted to \( [b_{i-1}, a_i] \) is a monotone mapping of \( [b_{i-1}, a_i] \) onto \( I \), and, for \( i = 0, 1, 2, \ldots, n \), \( f \) restricted to \( [a_i, b_i] \) is constant and equal to 0 or 1.

**Proof.** To prove the sufficiency of the stated condition suppose that \( n \) is a positive integer, \( 0 = a_0 \leq b_0 < a_1 \leq b_1 < \ldots < a_{n-1} \leq b_{n-1} < a_n \leq b_n = 1 \), and \( f : I \rightarrow I \) is a mapping satisfying the condition above with respect to \( a_0, a_1, \ldots, a_n \) and \( b_0, b_1, \ldots, b_n \). Suppose that \( K = [c, d] \) is a subinterval of \( I \), and that \( L \) is a component of \( f^{-1}(K) \). Suppose \( L \cap [b_{i-1}, a_i] \neq \emptyset \) for some \( i \). Since \( f \) is monotone on \( [b_{i-1}, a_i] \), there is a subinterval, \( J \), of \( [b_{i-1}, a_i] \) such that \( f(J) = K \). Also by the monotonicity of \( f \) on \( [b_{i-1}, a_i] \), \( J \cap L \neq \emptyset \). Since \( L \) is a component of \( f^{-1}(K) \), \( J \subset L \). Hence \( f(L) = K \). If \( L \cap [b_{i-1}, a_i] = \emptyset \) for all \( i \), then \( L \subseteq [a_i, b_i] \) for some \( i \). But, since \( L \) is a component of \( f^{-1}(K) \), \( L = [a_i, b_i] \) and thus \( L \cap [b_{i-1}, a_i] \neq \emptyset \), a contradiction. Therefore \( f \) is confluent.

Now suppose that \( f : I \rightarrow I \) is confluent. Suppose that \( f^{-1}(0) \) has \( n \) components, \( L_1 < L_2 \ldots < L_n \), and \( f^{-1}(1) \) has \( m \) components, \( H_1 < H_2 < \ldots < H_m \). Let \( k = \deg(f) = m + n - 1 \). Either \( 0 \in L_1 \) or \( 0 \in H_1 \) by Lemma 4. Suppose the former. Define \( a_0 \) and \( b_0 \) to be the right and left endpoints of \( L_1 \) respectively; \( L_1 = [a_0, b_0] \). For \( i > 0 \) define \( a_i \) and \( b_i \) to be the right and left endpoints, respectively, of \( H_{(i+1)/2} \) if \( i \) is odd, or to be the right and left endpoints, respectively, of \( L_{(i/2+1)} \) if \( i \) is even. We need only show that \( f \) is monotone on \( [b_{i-1}, a_i] \)
for $0 < i < k$. Suppose that $i > 0$ and that $i$ is odd. Let $j = (i + 1)/2$. Then $[a_{i-1}, b_{i-1}] = L_j$ and $[a_i, b_i] = H_j$. Suppose that $t$ is in $I$ and that $f^{-1}(t)$ has two components, $C_1$ and $C_2$, lying in $[b_{i-1}, a_i]$. By Lemma 2, there is a component of either $f^{-1}(0)$ or $f^{-1}(1)$ lying between $C_1$ and $C_2$. But $L_j$ and $H_j$ are consecutive components of $f^{-1}(0) \cup f^{-1}(1)$, so we have reached a contradiction. A similar contradiction arises if $i$ is even.

For mappings between Peano continua, light confluent mappings are open (see [6, Theorem 13.23]). The special case of this result where the continuum is $[0,1]$ follows as an easy corollary of Theorem 1.

**Corollary** If $f : I \to I$ is a light confluent mapping then $f$ is open.

**Proof.** Suppose that $f : I \to I$ is confluent. Let $n = \deg(f)$ and let $a_0 \leq b_0 < a_1 \leq b_1 < \ldots < a_n \leq b_n$ be the sequences given by Theorem 1. Since $f^{-1}(0)$ and $f^{-1}(1)$ are discrete, $a_i = b_i$ for $i = 1, 2, \ldots, n$. Since $f$ is monotone and light on $[a_{i-1}, a_i]$, $f$ is a homeomorphism from $[a_{i-1}, a_i]$ onto $I$ for $i = 1, 2, \ldots, n$. Thus, by [7, Lemma 1, p.453], $f$ is open.

**Theorem 2.** The uniform closure of the space of all open mappings from $I$ onto $I$ is the space of all confluent mappings from $I$ onto $I$.

**Proof.** Let $\mathcal{C}$ (respectively, $\mathcal{O}$) denote the space of all confluent (resp., open) mappings from $I$ onto $I$. Denote the supremum norm of a mapping $f : I \to I$ by $||f||$. We first note that $\mathcal{C}$ is uniformly closed. This can be seen from either [4, Theorem 5.48, p. 41] or [5, 3.1]. Thus, $cl(\mathcal{O}) \subset \mathcal{C}$. Suppose that $f : I \to I$ is confluent. Let $n = \deg(f)$ and let $a_0 \leq b_0 < a_1 \leq b_1 < \ldots < a_n \leq b_n$ be the sequences given by Theorem 1. Let $g : I \to I$ be the open mapping such that $g(a_0) = f(a_0)$, and $g(b_i) = f(b_i)$ for $1 < i < n$, and such that $g$ is linear on $[a_0, b_1]$ and on $[b_i, b_{i+1}]$ for $1 < i < n - 1$. For $k = 1, 2, 3, \ldots$ define $g_k : I \to I$ by

$$g_k(x) = \frac{1}{k}((k - 1)f(x) + g(x)).$$
Now $g$ is increasing on each of the intervals $[a_0, b_1], [b_1, b_2], \ldots, [b_{n-1}, b_n]$ on which $f$ is nondecreasing and $g$ is decreasing on each of those intervals on which $f$ is nonincreasing. Consequently $g_k$ is either increasing or decreasing on those same intervals. Thus $g_k$ is an open mapping for all $k$. Clearly

$$||g_k - f|| = \frac{1}{k}||g - f||,$$

so $\lim_{n \to \infty} g_k = f$ uniformly. Therefore $cl(\mathcal{O}) = C$.

A mapping which is the uniform limit of onto homeomorphisms is called a near-homeomorphism (see [1]). Likewise we define a mapping $f : X \to Y$ of continua to be near-open provided that it is the uniform limit of open mappings. With this terminology Theorem 2 may be restated as follows:

**Theorem 2a.** Every confluent mapping from $I$ onto $I$ is near-open.

**Lemma 6.** Suppose that $f : I \to I$ and $g : I \to I$ are confluent and $||f - g|| < \frac{1}{2}$.

1. If $f(a) = f(b) = 0$ then either $f$ is constant on $[a, b]$ or there is a number $c$ between $a$ and $b$ such that $g(c) = 1$.
2. If $f(a) = f(b) = 1$ then either $f$ is constant on $[a, b]$ or there is a number $c$ between $a$ and $b$ such that $g(c) = 0$.

**Proof.** To prove (1), suppose that $f$ is not constant on $[a, b]$. Then by Lemma 1 there is a number $c'$, such that $a < c' < b$, and $f(c') = 1$. Since $||f - g|| < \frac{1}{2}$, it follows that $g(c') > \frac{1}{2}$, $g(a) < \frac{1}{2}$, and $g(b) < \frac{1}{2}$. By the Intermediate Value Theorem, there are numbers $a'$ and $b'$ such that $a < a' < c' < b' < b$, $g(a') = \frac{1}{2}$, and $g(b') = \frac{1}{2}$. Then, by Lemma 1, there is a number $c$, with $a' < c < b'$, such that $g(c) = 1$.

**Lemma 7.** Suppose that $f : I \to I$ and $g : I \to I$ are confluent and $||f - g|| < \frac{1}{2}$.

1. Between consecutive components of $f^{-1}(0)$ there is just one component of $g^{-1}(1)$.
2. Between consecutive components of $f^{-1}(1)$ there is just one component of $g^{-1}(0)$. 
Proof. Suppose that $L_1$ and $L_2$ are consecutive components of $f^{-1}(0)$. By Lemma 6 there is at least one component of $g^{-1}(1)$ between $L_1$ and $L_2$. Suppose that $[a, b]$ and $[c, d]$ are components of $g^{-1}(1)$ which lie between $L_1$ and $L_2$ and that $b < c$. Then, by Lemma 6, there is a component of $f^{-1}(0)$ between $b$ and $c$, and consequently between $L_1$ and $L_2$. This is a contradiction.

Theorem 3. Suppose that $f : I \to I$ and $g : I \to I$ are confluent and that $\|f - g\| < \frac{1}{2}$. Then $\deg(f) = \deg(g)$.

Proof. Suppose that $f^{-1}(0)$ has $n_f$ components, $f^{-1}(1)$ has $m_f$ components, $g^{-1}(0)$ has $n_g$ components and that $g^{-1}(1)$ has $m_g$ components. Suppose that $f(0) = 0$. Since $\|f - g\| < \frac{1}{2}$ and $g$ is confluent, $g(0) = 0$. All components of $g^{-1}(1)$ lie in the complement of $f^{-1}(0)$, again since $\|f - g\| < \frac{1}{2}$. If $f(1) = 0$, then $g(1) = 0$ and all components of $g^{-1}(1)$ lie between components of $f^{-1}(0)$. Thus, in this case, $m_f = m_g$ by Lemmas 2 and 7. Now suppose that $f(1) = 1$. Let $[a, b]$ denote the right most component of $f^{-1}(0)$. There is at least one component of $g^{-1}(1)$ to the right of $b$ since $g(1) = 1$. If there were two components of $g^{-1}(1)$ to the right of $b$, by Lemma 7 there would be a component of $f^{-1}(0)$ to the right of $b$. Hence there is just one component of $g^{-1}(1)$ to the right of $b$. By similar reasoning, there is just one component of $f^{-1}(1)$ to the right $b$. All components of $g^{-1}(1)$ to the left of a lie between two components of $f^{-1}(0)$. Hence, by Lemmas 2 and 7, $g^{-1}(1)$ and $f^{-1}(1)$ have the same number of components to the left of $a$. Therefore $m_f = m_g$. Similar reasoning yields that $n_f = n_g$. Therefore $\deg(f) = \deg(g)$.

An inverse sequence is a pair $\{x_i, f_i\}$ whose first term is a sequence of spaces and whose second term is a sequence of mappings $f_i : X_{i+1} \to X_i$, called the bonding maps of the sequence. The inverse limit of the sequence $\{X_i, f_i\}$ is the set

$$\lim_{\to} \{X_i, f_i\} = \{(x_1, x_2, x_3, \ldots) | \ x_i \in X_i \text{ and } f_i(x_{i+1}) = x_i\},$$
with the topology induced by the metric
\begin{equation}
\bar{d}(\bar{x}, \bar{y}) = \sum_{i>0} \frac{d_i(x_i, y_i)}{2^i}
\end{equation}
where \( \bar{x} = (x_1, x_2, x_3, \ldots) \), and \( \bar{y} = (y_1, y_2, y_3, \ldots) \) are in \( \lim\{X_i, f_i\} \) and \( d_i \) is the metric for \( X_i \).

**Theorem 4.** Suppose that \( f_i : I \to I \) is confluent for \( i = 1, 2, \ldots \). Then there is a sequence \( g_i \) of open mappings from \( I \) onto \( I \) such that \( \deg(g_i) = \deg(f_i) \) and \( \lim\{I, f_i\} \) is homeomorphic to \( \lim\{I, g_i\} \).

**Proof.** For each \( i \) there exists a sequence \( \{h_{ij}\}_{j=1}^{\infty} \) of open mappings of \( I \) onto \( I \) which converges uniformly to \( f_i \) and which has the property that \( ||f_i - h_{ij}|| < \frac{1}{2} \) for all \( j \). By Theorem 3, \( \deg(f_i) = \deg(h_{ij}) \) for all \( i \) and \( j \). Let
\[ K_i = \{h_{ij}| j = 1, 2, \ldots \}. \]
By [1, Theorem 3, p.481], there is a sequence \( \{g_i\}_{i=1}^{\infty} \) such that \( g_i \) is in \( K_i \) and \( \lim\{g_i, I\} \) is homeomorphic to \( \lim\{f_i, I\} \).

If \( n \) is a positive integer, we define the **standard open mapping of degree \( n \)**, \( w_n : I \to I \), to be the mapping such that, for \( i = 1, \ldots n, \)
\begin{equation}
w_n(i/n) = \begin{cases} 
0 & \text{if } i \text{ is even} \\
1 & \text{if } i \text{ is odd},
\end{cases}
\end{equation}
and which is linear on the intervals \([ (i-1)/n, i/n ] \) (see [3,p. 203]).

**Corollary 1.** Suppose that \( f_i : I \to I \) is confluent for \( i = 1, 2, \ldots \) and that \( n_i = \deg(f_i) \). Then \( \lim\{f_i, I\} \) is homeomorphic to \( \lim\{I, w_{n_i}\} \).

**Proof.** This follows from Theorem 4 and [3, Lemma 4, p. 204].

**Corollary 2.** If \( f : I \to I \) and \( g : I \to I \) are confluent then \( \lim\{I, f\} \) and \( \lim\{I, g\} \) are homeomorphic if and only if \( \deg(f) \) and \( \deg(g) \) have the same prime factors.
Proof. This follows from Corollary 1 and [8, Thm. 5, p. 599].

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