PROXIMATELY REFINABLE MAPS AND 
$\theta'_n$-CONTINUA

by
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1. INTRODUCTION

A class of functions is of particular interest if many topological properties are preserved by the action of all of the functions. One class of functions that yields such stability for many continuum-theoretic properties is the class of refinable maps, defined by Jo Heath (Ford) and Jack Rogers [2, p. 263]. Proximate refinability, which generalizes refinability, was defined by Grace [4, p. 294] in considering what properties of refinable maps are essential to the preservation of the proximate fixed point property [4, p. 294]. One purpose of this paper is to characterize proximately refinable maps on graphs as those maps that are monotone. Another purpose is to study what properties of continua preserved by refinable maps are or are not preserved by proximately refinable maps where the domain or range is a \( \theta_n \)-continuum or a \( \theta'_n \)-continuum.

2. DEFINITIONS AND PRELIMINARIES

A continuum is a compact, connected, metric space and, except for Theorem 1, all spaces are continua. A \( \theta_n \)-continuum (defined in [6, p. 261] as a compact, metric \( \theta_n \)-space, based on the definition of \( \theta_n \)-space in [1, p. 139]) is a continuum in which the complement of any subcontinuum has at most \( n \) components. A \( \theta'_n \)-continuum [7, p. 56] is a \( \theta_n \)-continuum which admits a unique, monotone, upper semi-continuous decomposition \( D \), called the canonical decomposition, whose elements have void interior and whose quotient space \( X_D \) is a graph, i.e., a locally connected \( \theta_n \)-continuum. There is a
natural function \( P : X \to X_D \), called the projection function, that assigns to each element \( x \) in \( X \), the element of \( D \) that contains it. The elements of \( D \) are called tranches and the order of a tranche is its order considered as an element of the quotient space. If \( A \) is a subset or point of \( X \), then 
\[
K(A) = \bigcap \{Q \mid Q \text{ is a subcontinuum of } X \text{ and } A \text{ is contained in the interior of } Q\} \quad [9, \text{Theorem } 2, \text{p. } 404].
\]
It is easily seen that 
\[
K(H) = T(H) \quad \text{for each subcontinuum } H \text{ of a } \theta_n\text{-continuum }
\]
\( X \) where 
\[
T(H) = \{x \in X \mid K(x) \cap H \neq \emptyset\}.
\]
It also is easily seen that 
\[
K(H) \setminus H \text{ has void interior for any subcontinuum } H \text{ of a } \theta'_n\text{-continuum. From this and } [6, \text{Theorem } 1, \text{p. } 263], \text{it follows that a continuum } X \text{ is a } \theta'_n\text{-continuum if and only if } X \text{ is a } \theta_n\text{-continuum for which } K(H) \setminus H \text{ has void interior for each subcontinuum } H \text{ of } X.
\]

The notation \( f : X \to Y \) means \( f \) is a function from \( X \) onto \( Y \). A map is a continuous function (but the use of map as a verb does not require that the function be continuous). A function \( g \) is \( \varepsilon\)-continuous if, for each \( x \) in the domain of \( g \), there is a neighborhood \( D \) of \( x \) such that \( g[D] \subseteq N_\varepsilon(g(x)) \), i.e., such that 
\[
d(g(x), g(z)) < \varepsilon, \text{ for each } z \text{ in } D.
\]
A function \( f : X \to Y \) is an \( \varepsilon \)-function if \( \text{diam } (f^{-1}(y)) < \varepsilon, \text{ for each } y \text{ in } Y, \) and a strong \( \varepsilon \)-function if, for each \( y \) in \( Y \), there is a neighborhood \( D \) of \( y \) such that \( \text{diam } (f^{-1}[D]) < \varepsilon. \)
A function \( f : X \to Y \) is proximately refinable if, for every positive number \( \varepsilon \), there is an \( \varepsilon \)-continuous strong \( \varepsilon \)-function \( g : X \to Y, \) \( \varepsilon \)-near \( f, \) i.e., such that 
\[
d(f(x), g(x)) < \varepsilon \text{ for all } x \text{ in } X. \]
If \( g \) can be chosen to be continuous also, then \( f \) is refinable. It is easily seen that a proximately refinable or refinable function is a map. If \( f \) is refinable and \( g \) in the definition can be chosen to be one to one, then \( f \) is a near homeomorphism.

Let \( f : X \to Y \) be a proximately refinable map, and, for each positive integer \( n \), let \( f_n \) be a strong \( \frac{1}{n} \)-function that is \( \frac{1}{n} \)-continuous and \( \frac{1}{n} \)-near \( f \). Let \( B \) be a closed subset of \( Y. \) We denote by \( B' \) the limit of some convergent subsequence of \( \{f^{-1}_n[B]\}\). Observe that \( B' \) is not uniquely determined since it depends on the choice of the subsequence. Heath (Ford) and Rogers [2, Theorem 1, p. 264] have proved for refinable maps.
that for each subcontinuum $B$ of $Y$, $B'$ is a subcontinuum of $X$ such that $f[B'] = B$ and $f^{-1}[B^o] \subseteq B'$. Their proof [2, Theorem 1, p. 264] also works for proximately refinable maps. We will use both of these properties of $B'$ frequently.

3. PROXIMATELY REFINABLE MAPS ON GRAPHS AND COMPOSITION OF MAPPINGS

The first theorem in this section will be of use later in the section and is also of interest in its own right.

**Theorem 1.** Let $X, Y,$ and $Z$ be compacta and $f : X \to Y$ and $g : Y \to Z$ be proximately refinable. Then $h = g \circ f$ is proximately refinable, and $h$ is refinable, if $f$ and $g$ are.

**Proof.** Let $\epsilon$ be a positive number, and let $\delta < \epsilon$ be a positive number such that $d(g(y_1), g(y_2)) < \frac{\epsilon}{2}$, if $d(y_1, y_2) < \delta$. Let $F$ be a strong $\delta$-function that is $\delta$-continuous, and $\delta$-near $f$. Since strong $\delta$-functions are uniformly strong there is a positive number $\delta' < \frac{\epsilon}{2}$ such that $\operatorname{diam} (F^{-1}[D]) < \delta$, if $\operatorname{diam} D < \delta'$. Let $G$ be a strong $\delta'$-function that is $\delta'$-continuous and $\delta'$-near $g$. Let $H = G \circ F$ and let $D$ be any open subset of $Z$ such that $\operatorname{diam} (G^{-1}[D]) < \delta'$. Each point of $Z$ is contained in such an open set, since $G$ is a strong $\delta'$-function. So $\operatorname{diam} H^{-1}[D] = \operatorname{diam} (F^{-1}[G^{-1}[D]]) < \delta < \epsilon$, by the choice of $\delta'$. Hence, $H$ is a strong $\epsilon$-function.

Let $x \in X$. Then $d(f(x), F(x)) < \delta$. Therefore $d(g(f(x)), g(F(x))) < \frac{\epsilon}{2}$, also $d(g(F(x)), G(F(x))) < \delta' < \frac{\epsilon}{2}$. Hence $d(h(x), H(x)) = d(g(f(x)), G(F(x))) \leq d(g(f(x)), g(F(x))) + d(g(F(x)), G(F(x))) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Hence $H$ is $\epsilon$-near $h$. Since $h$ is continuous, $H$ is $(2\epsilon + \gamma)$-continuous, for each positive number $\gamma$, by [5, Lemma 2, p. 332]. It follows that $H$ is $3\epsilon$-continuous, and, hence, that $h$ is proximately refinable.

If $f$ and $g$ are refinable, then $F$ and $G$ can be chosen to be continuous (and still be strong $\delta$- and $\delta'$-functions, respectively). Then $H$ is continuous, and hence, $h$ is refinable.

Next we use Theorem 1 to establish a characterization of proximately refinable mappings on graphs.
Theorem 2. A map defined on a graph is proximately refinable if and only if it is monotone.

Proof. It has been proved in [5, Lemma 5, p. 335] that any proximately refinable map on a graph is monotone.

To show the converse, let $X$ be a graph and $f : X \to Y$ be monotone. Let $\{A_1, \ldots, A_n\}$ be the collection of all arcs and simple closed curves in $X$ having the following properties: (1) each is mapped to a point by $f$, (2) the end points of the arcs are not of order 2 but all other points are, and (3) all but one point of each simple closed curve is of order 2. Let $X_1$ be the space obtained from $X$ by identifying $A_1$ to a point and, in general, for $i = 2, \ldots, n$, let $X_i$ be the factor space obtained from $X_{i-1}$ by identifying $A_i$ in $X_{i-1}$ to a point. Note that an $A_i$ that starts as an arc may become a simple closed curve before it is shrunk to a point. Let $X_0 = X$ and for $i = 1, \ldots, n$, let $f_i$ be the natural map from $X_{i-1}$ to $X_i$. Then the map $f_{n+1} : X_n \to Y$ defined by $f_{n+1} = f \circ ([f_n \circ \cdots \circ f_1]^{-1})$ is monotone and has the property that $f_{n+1}^{-1}(y)$ contains at most one point of order unequal to 2 and contains no simple closed curve, for all $y$ in $Y$. Hence by [3, Theorem 2, p. 142], $f_{n+1}$ is proximately refinable (in fact, $f_{n+1}$ is a near homeomorphism). Also $f$ can be taken to be $f_{n+1} \circ f_n \circ \cdots \circ f_1$, and is proximately refinable by Theorem 1, if one can show that $f_i$ is proximately refinable for $i = 1, \ldots, n$. For notational convenience, we only show that $f_1 : X_0 \to X_1$ is proximately refinable. Also we show this only for the case where $A_1$ is an arc with endpoints of order greater than 2. The case where $A_1$ is an arc with one endpoint of order 1 and the other of order greater than 2 and the case where $A_1$ is a simple closed curve with all but one point of order 2 are similar.

Let $\epsilon$ be a positive number. Let the end points of $A_1$ be $b$ and $c$. Let $m + 1$ and $n + 1$ be the order of $b$ and $c$, respectively, in $X_0$. For $i = 1, \ldots, m$, and $j = 1, \ldots, n$, let $B_i$ and $C_j$ be arcs in $X_0 \setminus A_1$, of diameter less than $\frac{\epsilon}{2}$, that are mutually disjoint except that all of the $B_i$'s contain $b$ and all of the $C_j$'s contain $c$, and such that $b$ and $c$ are the only points in their union that
are not of order 2 in $X_0$. Since $A_1 \cup (\bigcup_{i=1}^m B_i) \cup (\bigcup_{j=1}^n C_j)$ is uniquely arcwise connected, no confusion should result from using interval notation for subarcs and half-open subarcs in that subgraph. Let $a_1$ be the identified point, $A_1$, as a point in $X_1$. Let $d_0$ be the distance in $X_0$ and $d_1$ be the distance in $X_1$. Assume $d_1(x,a_1) = d_0(x,A_1)$ and that $d_1(x,z) \leq d_0(x,z)$, for points in $X_1 \setminus \{a_1\} = X_0 \setminus A_1$.

For $i = 1, \ldots, m$ and $j = 1, \ldots, n$, let $b^i$ and $c^j$ be the end points of order 2 of $B_i$ and $C_j$, respectively. Let $b_0 = c_0$ be a point of $(b,c)$ and let $b_0, \ldots, b_p = b$ and $c_0, \ldots, c_q = c$ be partitions of $[b_0, b]$ and $[c_0, c]$, respectively, whose subarcs all have diameter less than $\frac{\varepsilon}{2m}$ and $\frac{\varepsilon}{2n}$, respectively, and such that $p$ is a multiple of $m$ and $q$ is a multiple of $n$. Let $u = \frac{p}{m}$ and $v = \frac{q}{n}$.

For $i = 1, \ldots, m$, and $j = 1, \ldots, n$ let $b = b_0^i, \ldots, b_{u+1}^i = b^i$ and $c = c_0^j, \ldots, c_{v+1}^j = c^j$ be partitions of $[b, b^i]$ and $[c, c^j]$, respectively.

We now wish to show that there is a strong $\varepsilon$-function, $f_\varepsilon$, that is $\varepsilon$-continuous and $\varepsilon$-near $f_1$. Let $f_\varepsilon$ be a function that maps $X_0$ onto $X_1$ in the following way. For $x \in X_0 \setminus [A_1 \cup (\bigcup_{i=1}^m B_i) \cup (\bigcup_{j=1}^n C_j)]$, let $f_\varepsilon(x) = f_1(x) = x$. For $i = 1, \ldots, m$ and $j = 1, \ldots, n$, let $f_\varepsilon$ map $(b_i, b_i]$ homeomorphically onto $(b_u^i, b^i]$ and map $(c, c^i]$ homeomorphically onto $(c_0^i, c^i]$. Let $f_\varepsilon(b_0) = a_1$. For $j = 1, \ldots, u$ and $k = 1, \ldots, m$, let $B(j, k) = (b_{j-1}^k, b_j^k]$. For $i = 1, \ldots, p$, let $f_\varepsilon$ map $(b_{i-1}, b_i]$ homeomorphically onto the $i$th member of $\{B(j, k) \mid j = 1, \ldots, u$ and $k = 1, \ldots, m\}$, in the lexicographical order on the ordered pairs $(j, k)$. Let $f_\varepsilon$ be defined on $[c_0, c]$ analogously. It is straightforward to check that $f_\varepsilon$ has the desired properties.

For $\theta'_n$-continua $X$ and $Y$ and a proximately refinable map $f$ from $X$ onto $Y$, the next theorem uses Theorem 2 to give information about $f_D$, the induced map on the graph decomposition spaces.

**Theorem 3.** If $X$ and $Y$ are $\theta'_n$-continua and $f : X \rightarrow Y$ is
proximately refinable, then $f$ induces a map $f_D$ from $X_D$ onto $Y_D$ and $f_D$ is proximately refinable.

**Proof.** Although [8, Theorem 3, p. 233] and [8, Theorem 4, p. 233] are stated for refinable maps, parts of their proofs can be applied to proximately refinable maps. For proximately refinable maps, the last paragraph of the proof of [8, Theorem 3] can be used to establish the existence of the induced map $f_D$ and the first paragraph of the proof of [8, Theorem 4] can be used to show that $f_D$ is monotone. By Theorem 2, then, $f_D$ is proximately refinable.

4. COMPARISONS WITH REFINABLE MAP RESULTS

If $X$ is a $\theta'_n$-continuum and $f : X \rightarrow Y$ is refinable, it is known that $Y$ is a $\theta'_n$-continuum [8, Theorem 3, p. 233]. However, for proximately refinable maps the most that can be said is given by the following theorem.

**Theorem 4.** If $f : X \rightarrow Y$ is proximately refinable and $X$ is a $\theta'_n$-continuum, then $Y$ is a $\theta_{2n}$-continuum.

**Proof.** Let $H_0$ be a subcontinuum of $Y$ such that $Y \setminus H_0 = H_1 \cup H_2 \cdots \cup H_m$, where $H_1, \ldots, H_m$ are mutually separated. For $i = 1, \ldots, m$, let $K_i = H_0 \cup H_i$. For $i = 1, \ldots, m$, let $f_i$ be a strong $\frac{1}{i}$-function that is $\frac{1}{i}$-continuous and $\frac{1}{i}$-near $f$ such that $\{f_i^{-1}[H_0]\}, \{f_i^{-1}[K_1]\}, \ldots,$ and $\{f_i^{-1}[K_m]\}$ converge to $H'_0$, $K'_1, \ldots,$ and $K'_m$, respectively. For $i = 1, \ldots, m$, $K'_i$ is contained in the component of $f^{-1}[K_i]$ that contains $f^{-1}[H'_0]$. Let $H_0^*$ be the component of $f^{-1}[H_0]$ that contains $H'_0$. By [6, Lemma 1, p. 262] and the fact that $P[X]$ is a graph, there are mutually disjoint open arcs $A_i \subseteq clP[K_i \setminus H_0^*]$ for $i = 1, \ldots, m$, which have one and only one end point in $P[H_0^*]$ and contain only points of order 2. Now $P[X]$ is clearly a $\theta_n$-continuum. Let $G$ be the graph resulting from shrinking $P[H_0^*]$ to a point in $P[X]$. Then $G$ is also a $\theta_n$-continuum and so, by [1, Theorem 49, p. 158], the order of $G$ at $P[H_0^*]$ (as a point of $G$) is not more than $2n$. But the order of $G$ at $P[H_0^*]$ is at least $m$, so $m \leq 2n$. Hence $Y$ is a $\theta_{2n}$-continuum.
The following example shows that \(2n\) in Theorem 4 cannot be reduced and also shows that \(Y\) need not be a \(\theta^*_m\)-continuum for any \(m\).

**Example 1.** For each natural number \(n\), a proximately refinable map from a \(\theta^*_n\)-continuum \(X\) onto a \(\theta^*_{2n}\)-continuum \(Y\) that is neither a \(\theta^*_{2n}\)-continuum nor a \(\theta^*_{2n-1}\)-continuum.

For \(n > 1\), the example is gotten by putting together copies of the example for the case \(n = 1\).

For \(n = 1\), let \(K\) be Knaster’s indecomposable continuum with two end points [10, Figure 5, p. 205] that is the union of semicircles described using numbers in the unit interval that have a base 5 representation not using “1” or “3.” The end points of \(K\) are \((0,0)\) and \((1,0)\).

Let \(C\) be the Cantor set. \(X\) is \(C \times K\), considered as a subset of \(R^3\), with the following identifications. \((0,0,0) = (1,0,0)\) and for each gap \((a,b)\) of \(C\), where \(b - a = \frac{1}{3n}\), \((a,0,0) = (b,0,0)\) if \(n\) is odd and \((a,1,0) = (b,1,0)\) if \(n\) is even. \(Y\) is \(\{0,1\} \times K\), but with \((0,0,0) = (1,0,0)\). For any point \((a,b,c)\) of \(X\), let \(f(a,b,c) = (0,b,c)\), if \(0 \leq a \leq \frac{1}{3}\), and let \(f(a,b,c) = (1,b,c)\), if \(\frac{2}{3} \leq a \leq 1\). That is (if we ignore the identifications) \(f\) projects each half of \(X\) parallel to the \(x\)-axis onto the nearest half of \(Y\). We take the distance, \(d\), on \(X\) (and \(Y\)) to be such that \(d((a,b,c),(a',b',c')) \leq \sqrt{(a-a')^2 + (b-b')^2 + (c-c')^2}\), with equality where \(a = a'\), i.e., where both points are on the same copy of \(K\).

\(X\) is easily seen to be a circularly chainable \(\theta^*_1\)-continuum, and \(Y\) is the union of two copies of \(K\) joined at an end point of each, and, therefore, is neither a \(\theta^*_2\)-continuum nor a \(\theta^*_1\)-continuum.

Let \(X_1\) be the left half of \(X\), i.e., let \(X_1 = \{(a,b,c) \in X \mid 0 \leq a \leq \frac{1}{3}\}\), and let \(Y_1 = \{0\} \times K\). We wish to define approximating functions for \(f \mid X_1\) in such a way that the symmetry of \(X\) can be used to extend them to all of \(X\) and to see that \(f\) is
proximately refinable. Let \( C_1, C_2, \ldots \) be a sequence of open chains covering \( \{0\} \times K \) in such a way that, for \( i = 1, 2, \ldots \),
(1) \( C_i \) chains \( Y_i \) from \( (0,0,0) \) to \( (0,1,0) \), (2) \( C_{i+1} \) refines \( C_i \) and runs straight through \( C_i \) then straight back through \( C_i \) and finally straight through \( C_i \) again, and (3) \( C_i \) is a \( \frac{1}{i} \)-chain with \( (0,0,0) \) only in the first link (in one natural way of defining the chains, \( C_i \) would, in fact, be a \( 1/(4 \cdot 5^{i-1}) \)-chain).

Let \( \epsilon \) be a positive number. For some pair \( i, j \) of natural numbers with \( i < j \), we use chainings of \( X_1 \) and \( Y_1 \), based on \( C_i \) and \( C_j \) to define a function \( f_{i,j} \), \( X_1 \rightarrow Y_1 \) such that \( f_{i,j} \) is a strong \( \epsilon \)-function that is \( \epsilon \)-continuous and maps each point \( x \) of \( X_1 \) into a member of \( C_i \) that contains \( f(x) \). The basic idea in defining \( f_{i,j} \) is to (1) chain \( Y_1 \) from \( (0,0,0) \) to \( (0,1,0) \) with a certain chain \( C_{i,j} \) that refines \( C_i \) and whose links are, in a certain sense, about as long as the links of \( C_i \) and as wide as the links of \( C_j \), (2) chain \( X_1 \) from \( (0,0,0) \) to \( (\frac{1}{3},1,0) \) with a certain chain \( L_{i,j} \), with the same number of links as \( C_{i,j} \) has, such that each link is the Cartesian product of a "subinterval" of \( C \) with the projection onto \( K \) of a link in \( C_i \), and (3) map each link of \( L_{i,j} \) onto the corresponding link of \( C_{i,j} \).

Let \( i \) be a natural number such that \( C_i \) is an \( \frac{\epsilon}{4} \)-chain. Let \( j \) be a natural number that is large enough (e.g., \( j > i + \lfloor \log_3 \frac{8}{\epsilon} \rfloor \)) to permit the following selection. Let \( \{x_0, x_1, \ldots, x_k\} \) be a partition of \( [0,\frac{1}{3}] \), with \( k = 3^{j-i} \), where (1) for \( p = 1, \ldots, k-1 \), the division point \( x_p \) is in a component \((a, b)\) of \([0,\frac{1}{3}] \setminus C \) where \((a,1,0) = (b,1,0)\), if \( p \) is odd, and \((a,0,0) = (b,0,0)\), if \( p \) is even, and (2) for \( p = 1, \ldots, k \), the diameter of \( C \cap [x_{p-1}, x_p] \) is less than \( \frac{\epsilon}{8} \). Note that \( C_j \) runs straight through \( C_i \) in one direction or the other a total of \( 3^{j-i} \) times. Let \( C_{i,j} \) be the chain consisting of all of the sets \( A \) such that \( A \) is the union of the members of some subchain of \( C_j \) that is maximal with respect to being contained in the same link of \( C_i \). Let \( L'_{i,j} = \{(C \cap [x_{p-1}, x_p]) \times A \mid p \in \{1, \ldots, k\} \text{ and } A \in C_i \} \). Let \( L_{i,j} \) be the chain (in \( X_1 \), from \( (0,0,0) \) to \( (\frac{1}{3},1,0) \)) consisting of all of the sets \( A \) such that \( A \) is the union of the members of some (one- or two-link) subchain of \( L'_{i,j} \) that is maximal
with respect to being mapped into the same member of \( C_i \) by \( f \). Most of the members of \( L_{i,j} \) are members of \( L'_{i,j} \) but, for example, if \( A \) is the last member of \( C_i \), then \((C \cap [x_0, x_1]) \times A\) and \((C \cap [x_1, x_2]) \times A\) are replaced by \((C \cap [x_0, x_2]) \times A\).

\( L_{i,j} \) is an open (relative to \( X_1 \)) \( \mathcal{C}^{2} \)-chain covering \( X_1 \), and the number of links in \( L_{i,j} \) is the same as the number in \( C_{i,j} \) and is \( m = (n - 1)k + 1 = (n - 1)3^{j-i} + 1 \), where \( n \) is the number of links in \( C_i \).

To define \( f_{i,j}^1 \), let \( L_{i,j} = \{L_1, \ldots, L_m\} \) and \( C_{i,j} = \{C_1, \ldots, C_m\} \). Let \( f_{i,j}^1 \) be any function from \( X_1 \) onto \( Y_1 \) such that \( f_{i,j}^1(x) \in C_p \) if and only if \( x \in L_p \) for \( p = 1, \ldots, m \), and (to facilitate extending \( f_{i,j}^1 \) to all of \( X \) \( f_{i,j}^1 \) carries more than one point onto \( f_{i,j}^1(\frac{1}{3}, 0, 0) \)). Since \( C_{i,j} \) is an \( \varepsilon \)-chain, \( f_{i,j}^1 \) is clearly \( \varepsilon \)-continuous. Since \( L_{i,j} \) is an \( \mathcal{C}^{2} \)-chain, \( f_{i,j}^1 \) is clearly a strong \( \varepsilon \)-function. Since \( C_i \) is an \( \varepsilon \)-chain, \( f_{i,j}^1 \) is clearly \( \varepsilon \)-near \( f \).

To extend \( f_{i,j}^1 \) to \( f_{i,j} \), defined on all of \( X \), let \( f_{i,j}(a, b, c) = f_{i,j}^1(a, b, c) \) if \( (a, b, c) \in X_1 \), and let \( f_{i,j}(a, b, c) \) be the point in \( Y \setminus Y_1 \) whose last two coordinates are the same as those of \( f_{i,j}^1(1 - a, b, c) \), if \( (a, b, c) \in X \setminus X_1 \). Now \( f_{i,j} \) carries \( X \) onto \( Y \), since \( f_{i,j} \) carries more than one point onto \( f_{i,j}^1(\frac{1}{3}, 0, 0) \), even though \( f_{i,j}(\frac{2}{3}, 0, 0) \not\in Y \setminus Y_1 \). From the corresponding properties of \( f_{i,j}^1 \), it is seen that \( f_{i,j} \) is an \( \varepsilon \)-continuous, strong \( \varepsilon \)-function that is \( \varepsilon \)-near \( f \). Hence \( f \) is proximately refinable.

For \( n > 1 \), let the domain be the union of \( n \) copies of \( X \) joined at the points that correspond to \( (0, 0, 0) \), let the image be the union of \( n \) copies of \( Y \), joined at the points that correspond to \( (0, 0, 0) \), and let the proximately refinable function be defined in the obvious way.

It is not known whether the condition that \( X \) is a \( \theta'_n \)-continuum can be weakened in Theorem 4, so we pose the following question.

**Question.** If \( X \) is a \( \theta_n \)-continuum and \( f : X \to Y \) is proximately refinable, must \( Y \) be a \( \theta_{2n} \)-continuum?
The reason the situation for proximately refinable maps differs from that for refinable maps, where the image of any $\theta_n'$-continuum is a $\theta_n'$-continuum [8, Theorem 3, p. 233], is as follows. A $\theta_n'$-continuum can be characterized as being a $\theta_n$-continuum in which $K(H) \setminus H$ has void interior for every subcontinuum $H$. If $f : X \to Y$ is refinable, and $X$ has this property (whether $X$ is a $\theta_n'$-continuum or not), then $Y$ has the property also [8, Corollary 2, p. 232]. However, as Example 1 shows, if $f$ is only proximately refinable, then $Y$ may not have this property. But if we add to the hypothesis of Theorem 4 the statement that $Y$ has this property, then we get the following theorem.

**Theorem 5.** Suppose $f : X \to Y$ is proximately refinable, $X$ is a $\theta_n$-continuum, and $Y$ is a continuum for which $K(H) \setminus H$ has void interior for each subcontinuum $H$ of $Y$. Then $Y$ is a $\theta_n'$-continuum.

**Proof.** Because $Y$ has the property, concerning the set function $K$, given in the hypothesis, to show that $Y$ is a $\theta_n'$-continuum it suffices to show that $Y$ is a $\theta_n$-continuum. Suppose $H_0$ is a subcontinuum of $Y$ and $Y \setminus H_0 = H_1 \cup \cdots \cup H_m$, where the $H_i$'s are mutually separated. For $i = 1, \ldots, m$, there is a proper subcontinuum $L_i$ of $Y$ such that $Y \setminus H_i \subseteq L_i$ since $K((Y \setminus H_i) \setminus (Y \setminus H_i))$ has void interior. Then $M_i = H_0 \cup (L_i \cap H_i)$ is a continuum that does not contain all of $H_i$ or any of $H_j$ for $j \neq i$. Let $M = \bigcup_{i=1}^m M_i$. Then $H_0 \subseteq M^0$ and $H_i \setminus M \neq \phi$, for $i = 1, \ldots, m$. Therefore, $f^{-1}[H_i] \setminus M' \neq \phi$, for $i = 1, \ldots, m$, since $M' \subseteq f^{-1}[M]$. Now $f^{-1}[H_0] \subseteq f^{-1}[M^0] \subseteq M'$, so $X = f^{-1}[Y] = f^{-1}((\bigcup_{i=1}^m H_i) \cup H_0) = (\bigcup_{i=1}^m f^{-1}(H_i)) \cup f^{-1}[H_0] = [\bigcup_{i=1}^m f^{-1}[H_i]] \cup M' = (\bigcup_{i=1}^m [f^{-1}(H_i) \setminus M']) \cup M'$. It follows that $M'$ separates $X$ into the $m$ nonvoid sets $f^{-1}[H_i] \setminus M'$, for $i = 1, \ldots, m$. Therefore $m \leq n$ since $X$ is a $\theta_n$-continuum. This proves that $Y$ is a $\theta_n$-continuum.

Unlike the inverses of refinable maps (see [8, Section 2, pp. 236–238]) the inverses of proximately refinable maps preserve almost nothing. The following theorem can be used to show that.
Theorem 6. Let $X = V \cup W \cup Z$ and $Y = V \cup W$, where $V, X, Y,$ and $Z$ are continua and $W$ is closed, and (1) $V$ is a chainable continuum with endpoint $p$, (2) for some neighborhood $N$ of $p, V \cap N \cap (W \cup Z) = \{p\}$, and (3) $Z \cap (V \cup W) = \{p\}$. Then the map $f : X \rightarrow Y$ that is the identity on $Y$ and maps $Z$ to $p$ is proximately refinable.

Proof. Let $\epsilon$ be a positive number. First we wish to cover $Z$ with a cover $A = \{A_1, \ldots, A_n\}$ of sets of cardinal $c$ such that (1) $A_1 \cup A_n$ is contained in the $\frac{\epsilon}{2}$-neighborhood of $p$, (2) $p \in A_1$, (3) $A_i \cap A_j = \phi$ unless $i = j$, (4) the diameter of $A_i \cup A_{i+1}$ is less than $\frac{\epsilon}{n}$, for $i = 1, \ldots, n - 1$, and (5) $n \geq 2$. Let $D$ be an open (relative to $Z$) $\frac{\epsilon}{4}$-cover of $Z$ (with subsets of $Z$) such that (1) no proper subcollection of $D$ covers $Z$, (2) $p$ is in only one member of $D$ and (3) $D$ has at least 2 members. Let $F$ be a function from some initial segment $\{1, \ldots, n\}$ of the natural numbers onto $D$, such that (1) $p \in F(1)$, (2) $F(i) \cap F(i + 1) \neq \phi$, for $i = 1, \ldots, n - 1$, and (3) $F(n) = F(1)$. $F$ can be defined by “walking” over $Z$ using members of $D$ as “stepping stones” and always “stepping” from one “stone” to an overlapping “stone,” starting with the “stone” covering $p$ and returning to that “stone” after “stepping” on all of the “stones.” For $i = 1, \ldots, n$, let $F_1(i)$ be the complement in $F(i)$ of the union of all of the other members of $D$.

We define $A$ by mathematical induction. Let $A_1$ be a subset of $F_1(1)$ such that $p \in A_1$ and $A_1$ and $F_1(1) \setminus A_1$ have cardinal $c$. Assume $A_i$ is defined for $i = 1, \ldots, j - 1$. If $F(j) = F(k)$ for some $k > j$, then let $A_j$ be a subset of $F(j) \setminus \bigcup_{i=1}^{j-1} A_i$ such that $A_j$ and $F_1(j) \setminus A_j$ have cardinal $c$. If $F(j) \neq F(k)$, for all $k > j$, then let $A_j = F(j) \setminus \bigcup_{i=1}^{j-1} A_i$. This defines $A$ having the desired properties.

Let $\{C_1, \ldots, C_m\}$ be an open (relative to $V$) $\frac{\epsilon}{2n}$-chain covering $V$ (with subsets of $V$) such that (1) $m > 2n$, (2) $\bigcup_{i=1}^{2n} C_i \setminus (W \cup Z) = \{p\}$, (3) $p$ is only in $C_1$ and (4) $C_i \cap C_j \neq \phi$, if and only if $|i - j| \leq 1$. Let $B_1 = C_1$ and $B_i = C_i \setminus C_{i-1}$, for $i = 2, \ldots, m$. Let $g$ be a one-to-one function from $X$ onto $Y$ such that (1) $g(A_i) = B_i$ and $g(B_{2i-1} \cup B_{2i}) = B_{n+i}$, for
$i = 1, \ldots, n$, and (2) $g$ is the identity on $W \cup \left[ \bigcup_{i=2n+1}^m B_i \right]$. Then $g$ is an $\epsilon$-continuous, strong $\epsilon$-function that is $\epsilon$-near $f$. It follows that $f$ is proximately refinable.

This can be used to show that inverses of proximately refinable maps do not preserve any of the properties considered in [8]. For example, for the property of being a $\theta'_1$-continuum, in the theorem let (1) $Z$ be the cone over the Cantor set, (2) $p$ be the vertex of $Z$, (3) $V$ and $W$ be arcs, each with end point $p$ and otherwise disjoint from $Z$, whose union is a simple closed curve, (4) $X = V \cup W \cup Z$, and (5) $Y = V \cup W$. By Theorem 6, the function $f : X \rightarrow Y$ that is the identity on $Y$ and maps $Z$ to $p$ is proximately refinable. $Y$ is a $\theta'_1$-continuum, but $X$ is not a $\theta_n$-continuum for any $n$, in fact, $\{p\}$ separates $X$ into $c$ components.

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