A DECOMPOSITION THEOREM FOR
\( \Sigma^* \)-SPACES

by

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Abstract. In this note it is shown that if $f$ is a continuous closed mapping from a $T_1, \Sigma^*$-space $X$ onto $Y$, then there is a $\sigma$-closed discrete subspace $Z$ of $Y$ such that $f^{-1}(y)$ is an $\omega_1$-compact subspace of $X$ for each $y \in Y \setminus Z$.

We assume that all spaces are $T_1$, and all mappings are continuous and onto.

In 1985, Y. Tanaka and Y. Yajima [4] obtained a decomposition theorem for $\Sigma$-spaces that every $\Sigma$-space $X$ satisfies the following condition (*).

(*) If $f$ is a closed mapping form $X$ onto $Y$, then there is a $\sigma$-closed discrete subspace $Z$ of $Y$ such that $f^{-1}(y)$ is an $\omega_1$-compact subspace of $X$ for each $y \in Y \setminus Z$.

J. Chaber [1] constructed a counterexample to show that $\Sigma^*$-spaces are not always satisfying the above condition (*). Since $\Sigma$-spaces are $\Sigma^*$-spaces, and $\Sigma^*$-spaces are $\Sigma^*$-spaces, it is a natural question whether $\Sigma^*$-spaces satisfy(*). Y. Tanaka and Y. Yajima [4] obtained only a weak form of decomposition theorem for $\Sigma^*$-spaces. The purpose of this note is to prove that $\Sigma^*$-spaces satisfy(*).

Recall basic definitions concerning $\Sigma^*$-spaces. Suppose that $P$ is a collection of subsets of a space $X$. $P$ is called hereditarily closure-preserving (abbr. HCP) if $\{H(P); P \in P\}$ is closure-preserving for every subset $H(P) \subseteq P \in P$. A space $X$ is called a $\Sigma^*$-space (or, strong $\Sigma^*$-space) [3] if there is a covering $\mathcal{K}$ of $X$ by closed countable compact subsets (or, closed compact

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subsets) and a $\sigma$-HCP collection $\mathcal{P}$ of closed subsets of $X$ such that whenever $K \subset U$ with $K \in \mathcal{K}$ and $U$ open in $X$, then $K \subset P \subset U$ for some $P \in \mathcal{P}$. The $\mathcal{P}$ is called a $\sigma$-HCP closed (mod $\mathcal{K}$)-network for $X$.

**Lemma (4, Lemma 1.1)** If $\mathcal{P}$ is an HCP collection of subsets of $X$, then

$$\{P_1 \cap P_2 \ldots \cap P_n; P_i \in \mathcal{P}, i \leq n\}$$

is also HCP in $X$ for each $N \in N$.

**Theorem** $\Sigma^*$-spaces satisfy $(\ast)$.

**Proof.** Suppose that $f$ is a closed mapping from a $\Sigma^*$-space $X$ onto $Y$. Let $\mathcal{K}$ be a covering of $X$ by closed countable compact subsets, and let $\mathcal{P} = \bigcup \{\mathcal{P}_n : n \in N\}$ be a $\sigma$-HCP closed (mod $\mathcal{K}$)-network for $X$, where each $\mathcal{P}_n$ is HCP in $X$. Here we can assume that $\mathcal{P}$ is closed under finite intersections by Lemma, and $X \in \mathcal{P}_n \subset \mathcal{P}_{n+1}$. For each $n \in N$, put

$$D_n = \{x \in X : \mathcal{P}_n \text{ is not point-finite at } x\},$$

then

1. $D_n$ is a $\sigma$-closed discrete subspace of $X$. In fact, for each $m \in N$, put

$$E_m = \{x \in X : \cap \{P \in \mathcal{P}_m : x \in P\} = \{x\}\},$$

then $E_m$ is a closed discrete subspace of $X$ by [3, Lemma 2.5]. It is not difficult to check that $D_n \subset \bigcup \{E_m : m \in N\}$ (cf.[2]), hence $D_n$ is $\sigma$-closed discrete in $X$. Put

$$\mathcal{Q}_n = \{P \setminus D_n : P \in \mathcal{P}_n\},$$

and

$$\mathcal{Q} = \bigcup \{\mathcal{Q}_n : n \in N\},$$

then

2. There are $m \geq n, F \in \mathcal{Q}_m$, and $G \subset D_m$ with $\cap \mathcal{F} = F \cup G$ for each finite $\mathcal{F} \subset \mathcal{Q}$ and $n \in N$.

In fact, let $\mathcal{F} = \{F_i : i \leq k\} \subset \mathcal{Q}$, we might as well grant $\cap \mathcal{F} \neq \emptyset$, and there are $n_i \in N, P_i \in \mathcal{P}_{n_i}$ with $F_i = P_i \setminus D_{n_i}$ and $n_i \leq n_{i+1}$, then $\cap \mathcal{F} = \cap \{P_i : i \leq k\} \setminus D_{n_k}$, and $\cap \{P_i : i \leq k\} \setminus D_{n_k}$.
\[ k = P \text{ for some } m \geq \max \{ n_k, n \}, P \in \mathcal{P}_m \] because \( \mathcal{P} \) is closed under finite intersections. Put
\[
F = P \setminus D_m, \quad \text{and} \quad G = P \cap (D_m \setminus D_{n_k}),
\]
then \( F \in \mathcal{Q}_m, G \subseteq D_m \) and \( \cap \mathcal{F} = F \cup G. \)

For each \( n \in \mathbb{N}, \) put
\[
Z_n = f(D_n) \cup \left( \bigcup \{ f(Q) \cap f(Q') : Q, Q' \in \mathcal{Q}_n, \right.
\]
\[
\left. \text{and } F(Q) \cap f(Q') \text{ is finite} \}\right).
\]

Since \( \mathcal{Q}_n \) is HCP in \( X, \) \( \{ f(Q) : Q \in \mathcal{Q}_n \} \) is HCP in \( Y. \) Thus \( Z_n \) is \( \sigma \)-closed discrete in \( Y \) by (1) and Lemma. Put
\[
Z = \cup \{ Z_n : n \in \mathbb{N} \},
\]
then \( Z \) is \( \sigma \)-closed discrete in \( Y. \) Take a \( y \in Y \setminus Z, \) then
\[(3) \{ Q \in \mathcal{Q}_n : Q \cap f^{-1}(y) \neq \emptyset \} \text{ is finite.} \]

Assume the contrary, then there is an \( m \in \mathbb{N} \) and a sequence \( \{ Q_n \} \) of distinct members of \( \mathcal{Q}_m \) such that \( Q_n \cap f^{-1}(y) \neq \emptyset. \)

Pick an \( x \in f^{-1}(y), \) then \( x \notin X \setminus D_n \) for each \( n \in \mathbb{N}, \) put
\[
R_n = \cap \{ Q \in \mathcal{Q}_n : x \in Q \}. \]

Since \( \mathcal{Q}_n \) is point-finite on \( X \) there are a \( k_n \in \mathbb{N}, F_n \in \mathcal{Q}_{k_n} \)
and \( G_n \subseteq D_{k_n} \) with \( m \leq k_n < k_{n+1} \) and \( R_n = F_n \cup G_n \) by (2).

Put
\[
F'_n = Q_n \setminus D_{k_n}, \quad G'_n = Q_n \cap D_{k_n},
\]
then \( F'_n \in \mathcal{Q}_{k_n}, G'_n \subseteq D_{k_n}, \) and \( Q_n = F'_n \cup G'_n. \) Since \( y \in f(R_n) \cap f(Q_n) \setminus Z = f(F_n) \cap f(F'_n) \setminus Z, f(F_n) \cap f(F'_n) \) is an infinite set. So we can choose a sequence \( \{ y_n \} \) of distinct points in \( Y \)
such that \( y_n \in f(F_n) \cap f(F'_n). \) Pick \( p_n \in F_n \cap f^{-1}(y_n), \) and \( q_n \in F'_n \cap f^{-1}(y_n). \)

Suppose that the sequence \( \{ p_n \} \) has not any cluster point in \( X. \) Take a \( K \in \mathcal{K} \) with \( x \in K, \) then there is an \( i \in \mathbb{N} \) such that
\( K \cap \{ p_n : n \geq i \} = \emptyset, \) thus \( x \in K \subseteq P \subseteq X \setminus \{ p_n : n \geq i \} \) for some \( j \geq i, P \in \mathcal{P}_j. \) Since \( y \in Y \setminus Z, x \notin D_j, \) then \( x \in P \setminus D_j \in \mathcal{Q}_j, \) hence \( R_j \subseteq P \setminus D_j, \) so \( p_j \in F_j \subseteq R_j \subseteq P \setminus D_j \subseteq X \setminus \{ p_n : n \geq i \}, \) a contradiction. Consequently, the sequence \( \{ p_n \} \) has a cluster point in \( X, \) and the sequence \( \{ y_n \} \) also has a cluster.
point in $Y$. On the other hand, since $q_n \in F'_n \subset Q_n \in Q_m$ for each $n \in N$, the sequence $\{q_n\}$ has not any cluster point in $X$ so the sequence $\{y_n\}$ has not any cluster point in $Y$ either, a contradiction. (3) holds. By

$$y \in Y \setminus Z, f^{-1}(y) \subset X \setminus \{D_n : n \in N\},$$

thus

$$\{f^{-1}(y) \cap P : P \in \mathcal{P}\} = \{f^{-1}(y) \cap Q : Q \in \mathcal{Q}\},$$

therefore it is a countable closed (mod $\mathcal{K}'$)-network from (3), where $\mathcal{K}' = \mathcal{K}_{f^{-1}(y)}$. Hence $f^{-1}(y)$ is an $\omega_1$-compact subspace of $X$ by [4, Lemma 1.2].

**Corollary** If $f$ is a closed mapping from a strong $\Sigma^*$-space $X$ onto $Y$, then there is a $\sigma$-closed discrete subspace $Z$ of $Y$ such that $f^{-1}(y)$ is Lindelöf for each $y \in Y \setminus Z$.

**Proof.** An $\omega_1$-compact strong $\Sigma^*$-space is a space with a countable (mod $\mathcal{K}$)-network with respect to $\mathcal{K}$ by compact subsets. A space with a countable (mod $\mathcal{K}$)-network with respect to $\mathcal{K}$ by compact subsets is Lindelöf.

**References**


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