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NEW CLASSIC PROBLEMS

by

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NEW CLASSIC PROBLEMS

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Mary Ellen Rudin and Frank Tall organized a problem session at the Spring Topology Conference in San Marcos, Texas in 1990 and invited several people to come up with their ideas for problems that should be the worthy successors to the S & L problems, the box product problems, the normal Moore space problem, etc. in the sense that they could and should be the focus of common activity during the 1990's as the older problems had been during the 1970's. They hoped that these problems would counterbalance the more centrifugal 1980's, during which there was a tendency for each set-theoretic topologist to do his own thing, rather than there being many people working on problems generally recognized as important. This compilation is the result. Time will tell whether the title is appropriate.

A PROBLEM OF KATETOV

Zoltan Balogh

Given a topological space X, let Borel (X) and Baire (X) denote the σ-algebras generated by the families closed $(X) = \{F : F \text{ is a closed set in } X\}$ and zero $(X) = \{Z : Z \text{ is a zero-set in } X\}$, respectively. The following question is due, without the phrase "in ZFC", to M. Katetov [K]:

Problem. *Is there, in ZFC, a normal T_1 space X such that Borel (X) =Baire (X) but X is not perfectly normal (i.e., closed $(X) \neq$ zero (X))?*

Related Problems *What if X is also locally compact? First countable? Hereditarily normal?*

Partial Results. There are several consistency examples [B]. CH implies that there is a locally compact, locally countable X satisfying the conditions of the problem. The existence of a first countable, hereditarily paracompact X is consistent, too. However, a space giving a positive answer to the problem cannot be any of the following: compact (Halmos); submetacompact and locally compact (Burke), Lindelöf and Čech-complete (Comfort), a subparacompact $P(\omega)-$ space (Hansell).

[K] M. Katetov, *Measures in fully normal spaces*, Fund. Math. **38** (1951), 73-84.

[B] Z. Balogh, *On two problems concerning Baire sets in normal spaces*, Proc. Amer. Math. Soc. **103** (1988), 939-945.

QUESTIONS

S.W. Davis

Question 1. *Is there a symmetrizable Dowker space?*

If X is such a space, then let $< F_n : n \in \omega >$ be a decreasing sequence of closed sets with $\cap_n F_n = \emptyset$ which can not be "followed down" by open sets, then attach $x_\infty \notin X$ to X and extend the symmetric so that $B(x_\infty, \frac{1}{n}) = F_n$, and the resulting space has a point, x_∞, which is not a G_δ−set. This answers the following old question of Arhangel'skiĭ and Michael.

Question 2. *Is every point of a symmetrizable space a G_δ−set?*

Results.

[Davis, Gruenhage, Nyikos, Gen. Top. Appl. 1978]

1. There is a T_3, zero dimensional symmetrizable space with a closed set which is not a G_δ. (Also not countably metacompact).

2. There is a T_2 symmetrizable space with a point which is not a G_δ. (Constructed as above).

3. In the example of 2., the sequential order, $\sigma(X)$, is 3.

4. If X is T_2 symmetrizable and $\sigma(X) \leq 2$, then each point of X is a G_σ–set.

[R.M. Stephenson, Can. J. Math. 1977, Top. Proc. 1979]

1. If X is T_2 symmetrizable and $x \in X$ is not a G_δ–set, then $X\backslash\{x\}$ is not countably metacompact.

2. If X is a regular feebly compact space which is not separable, then X has a point which is not a G_δ–set.

[Burke, Davis, Pac. J. Math. 1984, Top. Proc. 1980]

1. $\underline{b} = \underline{c} \rightarrow$ Every regular symmetrizable space with a dense conditionally compact subset is separable.

2. $\underline{b} = \underline{c} \rightarrow$ Every feebly compact regular symmetrizable space with a dense set of points of countable character is first countable.

3. If X is T_2 symmetrizable $\mathrm{cf}(\kappa) > \omega$ and $\chi(x, X) \leq \kappa$, then $\psi(x, X) < \kappa$. Hence, an absolute example must be non-separable and in fact have $\chi(X) > c$.

[Tanaka, Pac. J. Math. 1982, private communication 1984]

1. There is a regular symmetrizable X with $\chi(X) > c$. However, this example is perfect.

Question 3. *Is there a symmetrizable L-space?*

Results.

[Nedev, Soviet Math. Dokl. 1967]

1. Lindelöf symmetrizable spaces are hereditarily Lindelöf.

2. No symmetrizable L-space can have a weakly Cauchy symmetric.

[Kofner, Mat. Zametki, 1973][Davis, Proc. Amer. Math. Soc. 1982, Top Proc. 1984]

1. No symmetric L-space can have a structure remotely resembling a weakly Cauchy symmetric.

[Juhasz, Nagy, Szentmiklossy, C.R. Bul. Acad. Sci., 1985]

1. $CH \rightarrow$ there is a T_2, non-regular, symmetrizable space which is hereditarily Lindelöf and non-separable.

[Shakhmatov, C.R. Bul. Acad. Sci., 1988]

1. There is a model which contains a regular symmetrizable L-space.

[Balogh, Burke, Davis, C.R. Bul. Acad. Sci. 1989]

1. ZFC $\rightarrow$ there is a T_2 non-regular, symmetrizable space which is hereditarily Lindelöf and non-separable.
2. There is no left separated Lindelöf symmetrizable space of uncountable cardinality.

QUESTIONS

Alan Dow

1 Remote Points

A point $p \in \beta X \backslash X$ is a *remote* point of X if p is not in the closure of any nowhere dense subset of X. It is known that pseudocompact spaces do not have remote points[10,5] and that not every non-pseudocompact space has a remote point [12]. Every non-pseudocompact metric space has remote points [2] (or of countable π-weight [11]), but the statement "every non-pseudocompact space of weight $\aleph_1$ has remote points"" is independent of ZFC ([9,3]). There is a model in which not all separable non-pseudocompact spaces have remote points [5]. It follows from CH that all non-pseudocompact ccc spaces of weight at most $\aleph_2$ have remote points[7].

Question 1. *Does it follow from* CH *(or is it consistent with* CH*) that if some non-empty open subset of a non-pseudocompact space X is ccc then X has remote points?*

Question 2. *Is there a compact nowhere ccc space X such that $\omega \times X$ has remote points?*

Question 3. *Is there, for every space X, a cardinal κ such that $\kappa \times X$ has remote points?* (It is shown in [6] that a "no" answer implies the consistency of large cardinals.)

2 The uniform ultrafilters

Question 4. *Are there weak P_{ω_2}–points in $U(\omega_1)$ - the space of uniform ultrafilters on ω_1?*

A point $p \in X$ is a *weak* P_κ-point if p is not a limit point of any subset of X of cardinality less than κ. Kunen[8] constructed weak P_{ω_1}-points in $U(\omega)$. In fact Kunen introduced the notion of κ-OK points, showed that an ω_1-OK point is a weak P_{ω_1}–point and constructed 2^ω-OK points in $U(\omega)$. Unfortunately, it is not true that a 2^{ω_1}-OK point is a weak P_{ω_2}–point, (ω_2-OK points of $U(\omega_1)$ are constructed in [4]). However Kunen does show that a κ^{++}-GOOD point is a weak P_κ-point (a proof is given in [4]). The following show that it is consistent to suppose that there are ω_2-good points (hence weak P_{ω_2}-points) in $U(\omega_1)$.

A point $p \in X$ is κ-good if, for every function g from $[\kappa]^{<\omega}$ to the neighbourhood filter $\mathcal{F}$ of p, there is another function f from κ to $\mathcal{F}$ such that $\cap_{\alpha \in S} f(a) \subset g(S)$ for each $S \in [\kappa]^{<\omega}$. A point is κ-OK if the above holds when it is assumed that $g(S) = g(T)$ for all $S, T \in [\kappa]^{<\omega}$ with $\mid S \mid = \mid T \mid$. Recall that $\mathbf{GMA}$($\aleph_1$-centered) (see [14]) is the statement: Suppose P is a partial order which is $\aleph_1$-centered and every countable centered subset has a lower bound, then for any family $\mathcal{D}$ of fewer than $2^{\aleph_1}$ dense open subsets of P, there is a $\mathcal{D}$-generic filter $G \subset P, i.e. G \cap D \neq \emptyset$ for each $D \in \mathcal{D}$.

Now suppose that $\mathcal{F} \subset [\omega_1]^{\omega_1}$ is a filter such that $\mid \mathcal{F} \mid < 2^{\aleph_1}$ and let $g : [\omega_2]^{<\omega} \mapsto \mathcal{F}$. Let $P = \{p$: there is a countable

$I_p \subset \omega_2$ and an $\alpha_p < \omega_1$ such that $p : I_p \times \alpha_p \mapsto 2\}$ and let $p < q$ if $p \supset q$ and for any $S \in [I_q]^\omega g(S) \supset \{\beta > \alpha_q : (\forall \xi \in S) p(\xi, \beta) = 1\}$. It is easy to see that every countable centered subset of P has a lower bound (the union). For each $\alpha \in \omega_1$, let $P_\alpha = \{p \in P : \alpha_p = \alpha\}$. It follows from CH (hence from **GMA** ($\aleph_1$–centered)) that P_α is $\aleph_1$-centered for each α. Finally, for $F \in \mathcal{F}, \beta \in \omega_1$, and $S \in [\omega_1]^\omega$, let $D(F, \beta, S) = \{p \in P : (\exists \alpha \in (F \cap g(S)) \backslash \beta)(\forall \xi \in S) p(\xi, \alpha) = 1\}$. Each such $D(F, \beta, S)$ is dense open in P. It then follows from **GMA**($\aleph_1$-centered) $+2^{\omega_1} > \omega_2$ that there is a filter $\mathcal{F}' \subset [\omega_[1]^{\omega_1}$ and a function $f : \omega_2 \mapsto \mathcal{F}'$ such that $\mathcal{F}' \supset \mathcal{F}$ and $g(S) \supset \cap_{\xi \in S} f(\xi)$ for each $S \in [\omega_2]^{<\omega}$.

Question 5. *Do there exist points $p, q \in U(\omega_1)$ such that there are embeddings f, g of $\beta\omega_1$ into $\beta\omega_1$ with $f(p) = g(q)$, but no embedding takes p to q or q to p?*

A "yes" answer to Question 5 would imply that $\beta\omega_1$ fails to have the Frolík property (introduced in [1]).

3 Subspaces of ED spaces

Question 6. *Does there exist a compact zero-dimensional F-space (or basically disconnected space) which cannot be embedded into an extremally disconnected (ED) space?*

A "yes" answer is shown to be consistent with ZFC in [13]. However the problem is even open under CH and the basically disconnected question is completely open.

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HOMOGENEITY OF X^∞

G. Gruenhage

Problem. *is X^∞ homogeneous for every 0-dimensional first-countable regular space X? What if X is compact? What if X is a 0-dimensional subspace of the real line?*

D.B. Motorov[Mo] has shown that if X is 0-dimensional, first-countable, and compact, the X^∞ is a retract of a homogeneous space. G. Gruenhage and H. Zhou have shown (unpublished) that a 0-dimensional first-countable space is homogeneous if it contains a dense set of isolated points.

Concerning the metric case, the following result was announced by S.V. Medvedev[Me]. (F. van Engelen obtained this result independently.)

Theorem. *Let X be a metrizable space with dim $X = 0$. If X is first category, or X contains a dense absolute G_δ−set, then X^∞ is homogeneous.*

Corollary *Let X be a 0-dimensional subset of the real line. Then X^{∞} is homogeneous in the following cases:*
(a) X is analytic;
(b) $|X| < c$ and Martin's Axiom holds.

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DICHOTOMIES IN COMPACT SPACES AND T_5 SPACES

Peter J. Nyikos

Back in 1976-77, I wrote about eight "classic problems' in this Problem Section (v.1 and v.2). The first one was "Efimov's Problem", whose negation reads:

Problem 1. *Is there an infinite compact T_2 space which contains neither a nontrivial convergent sequence nor a copy of $\beta\omega$?*

Remarkably little progress has been made on this problem since then. The consistency result which subsumes all others is: "Yes if $s = \omega_1$" and was already essentially known back then: the CH construction by Fedorchuk in [1] and his PH construction in [2] cover all the bases. Indeed, all that is needed to get "no convergent sequences" in the latter construction is $s = \omega_1$ and the space is of character ω_1 so that under $\neg CH$ it cannot contain a copy of $\beta\omega$.

Also perhaps surprisingly, we know even less about the zero-dimensional case. The $s = \omega_1$ construction is zero-dimensional, but the only zero-dimensional construction of a CH-compatible example for Problem 1 that I know of is the $\diamondsuit$ one by Fedorchuk in [3].

The zero-dimensional version is equivalent to:

Problem 2 *Is there an infinite Boolean algebra (BA) which has neither a countably infinite homomorphic image nor a complete infinite homomorphic image?*

The equivalence is an elementary exercise in Stone duality. A compact zero-dimensional space contains a nontrivial convergent sequence iff it has an infinite closed metrizable subspace, and is metrizable iff its BA of clopen sets is countable. A BA is complete iff its Stone space is extremally disconnected, and every infinite compact extremally disconnected space contains a copy of $\beta\omega$ [4]: in fact it is enough to assume "basically disconnected," so that in Problem 2 one can equally well substitute " countably complete" for "complete": in other words, a $\sigma-$ algebra. Also equivalent is:

Problem 2′ *Is there an infinite BA which has neither a countably infinite homomorphic image nor an independent subset of cardinality c?*

The equivalence follows from the folklore result that a compact T_2 space can be mapped onto $[0,1]^c$ iff it contains a copy of $\beta\omega$. (To go one way, use the theory of absolutes: since $[0,1]^c$ contains a copy of $\beta\omega$, so does any compact T_2 space mapping onto it. To go the other way, use the separability of $[0,1]^c$ and the Tietze-Urysohn extension theorem.) This suggests replacing "independent subset of cardinality c" in Problem 2′ with "uncountable independent subset," as well as:

Problem 3. *Is there an infinite compact T_2 space which cannot be mapped onto $[0,1]^{\omega_1}$ and in which every convergent sequence is eventually constant?*

For more on the theme of mapping onto $[0,1]^\kappa$, see [5]. There in particular one sees that it is equivalent to having a closed subspace in which every point is on $\phi-$ character $\geq \kappa$.

This brings to mind another set of problems featuring a dichotomy of subspaces, on which there has been much recent activity and substantial progress. Though they are similar on the surface to the foregoing ones, the consistency results all pull in the opposite direction, so I will state them accordingly. One is often referred to as "Husek's problem":

Problem 4. *Does every infinite compact T_2 space contain either a nontrivial convergent ω—sequence or a nontrivial convergent ω_1-sequence?*

Recently Szentmiklóssy and Juhász have shown [6] that every compact T_2 space of uncountable tightness contains a convergent ω_1-sequence which is a free sequence, so that the answer is affirmative under PFA, which implies every compact of countable tightnss is sequential [7]. They also obtain an affirmative answer assuming ♣. Of course the answer is also affirmative under CH, since (easy exercise) every point of character ω_1 in a compact T_2 space has a nontrivial ω_1-sequence converging to it. Hence the following problem, posed by Juhász about the same time (late seventies) that Husek's was, also has an affirmative answer under CH:

Problem 5. *Does every infinite compact T_2 space contain either a point of first countability or a convergent ω_1-sequence?*

Szentmikllóssy and Juhász also show that the answer to this problem is affirmative if any number of Cohen reals are added to a model of CH, drawing upon some earlier results of Alan Dow. The answer is also affirmative under PFA since, as Dow showed, PFA implies every compact space of countable tightness has a point of first countability.

Another strengthening of Problem 4 is:

Problem 6. *Does every infinite compact T_2 space have a closed subspace with a nonisolated point of character $\leq \omega_1$?*

A convergent free ω_1-sequence together with its limit point is such a subspace, so again PFA implies an affirmative answer. Incidentally, this has as a corollary the fact that PFA implies every infinite compact T_2 space contains either the one-point compactification of a discrete space of size ω or the one-point Lindelöfization of a discrete space of size ω_1! And this is an independence result since Fedorchuk's $\diamondsuit$ examples in [3] contain neither. But there is another interesting dichotomy in [6] assuming ♣ (which is implied by $\diamondsuit$): then every compact T_2 space contains either the one-point compactificiation of a discrete space of size ω or the one-point Lindelöfization of an S-space in which every open subset is either countable or co-countable.

The BA versions of the zero-dimensional cases of Problems 4 and 6 are well-known, and can be tersely stated:

Problem 7. *Is every infinite BA of altitude $\leq \omega_1$ of pseudo-altitude $\leq \omega_1$?*

For information on these problems, see [8] and [9]. The former also gives a wide range of forcing models in which Problem 5 has an affirmative answer for separable zero-dimensional spaces, while the latter has a more restrictive family of models in which every infinite separable compact zero-dimenional space either admits a continuous map onto $[0,1]^{\omega_1}$ or has a point of character $\leq \omega_1$.

On the other hand, an affirmative answer to the following question would solve Problem 6 and hence the second half of Problem 7 negatively and thus establish the independence of these statements:

Problem 8. *Is MA + $\neg$CH (or even $p > \omega_1$) compatible with the existence of an infinite compact space T_2 of countable tightness with no nontrivial convergent sequences?*

Indeed, if a point of character $< p$ in a compact space of countable tightness is in the closure of a subset A, then there is a sequence from A converging to that point.

In the original version of this paper, I had a number of problems on countably compact hereditarily normal (T_5) spaces: whether it was consistent that they be all sequentially compact, and that all separable ones are compact. These were answered affirmatively when Velićković showed that OCA implies there is no T_5 γN. Here are some problems on T_5 spaces which are still unsolved.

Problem 9. Is there a ZFC example of a separable, T_5, locally compact space of cardinality $\aleph_1$?

Of course, CH is equivalent to **R** being such a space. We will now show that if there is a space as in Problem 9, there is one that is locally countable, hence (by local compactness) first countable and scattered. If CH hold, there simply is such a space: then Kunen "line" is an example. If $q > \omega_1$, then the Cantor tree over a Q-set is an example

If CH fails, then any space X as in Problem 9 is scattered, as is any (locally) compact space of cardinality $< c$. In the case where $q = \omega_1 < c$. Such a space X, if it exists, will have a locally countable open subspace of cardinality $\aleph_1$. This is shown by an analysis of the Cantor-Bendixson levels $X_{(\alpha)} = X^{(\alpha+1)} \backslash X^{(\alpha)}$, where $X^{(0)} = X$ and $X^{(\alpha+1)}$ is the derived set of $X_{(\alpha)}$. Obviously $X_{(0)}$ is countable, and so each point of $X_{(1)}$ has a countable neighborhood, whence $X_{(0)} \cup X_{(1)}$ is first countable. Now $q = \omega_1$ implies every separable first countable T_5 space is of countable spread, so $X_{(1)}$ is countable, and so we proceed by induction to $X_{(\alpha)}$ for each countable α. Now $\cup\{X_{(\alpha)} : \alpha < \omega_1\}$ is locally compact, locally countable, and has cardinality $\aleph_1$.

It is also an S-space! This is because it can be right-separated in order type ω_1 and is of countable spread. But we do not know whether $q = \omega_1$ is enough to produce an S-space, let alone one that is locally compact, locally countable, and T_5. And this is what Problem 9 boils down to:

Problem 9′. *Is there a locally compact, locally countable, T_5 S-space in every model of $q = \omega_1$?*

Problem 9 is also equivalent to:

Problem 9″. *Is there a ZFC example of a separable, T_5, locally compact, uncountable scattered space?*

We have already seen one implication. For the other, suppose Y is as in Problem 9″. If $Y_{(\alpha)}$ is countable for all countable α, then $\cup\{Y_{(\alpha)} : \alpha < \omega_1\}$ is as in Problem 9. Otherwise, take the first α such that $Y_{(\alpha)}$ is uncountable, let Z be a subset of $Y_{(\alpha)}$ of cardinality $\aleph_1$, and then $\cup\{X_{(\beta)} \cup Z : \beta < \alpha\}$ is as in Problem 9.

A similar analysis can be done for:

Problem 10. *Is there a ZFC example of a separable, uncountable, scattered T_5 space?*

In this problem, the role of $q = \omega_1$ is taken over by $2^{\aleph_0} < 2^{\aleph_1}$: by Jones' Lemma, it is implies that any example X for Problem 10 is of countable spread, and $\cup\{X_{(\alpha)} : \alpha < \omega_1\}$ will then be a locally countable S-space. If $2^{\aleph_0} = 2^{\aleph_1}$, there is an old example of a space as in Problem 10, due to R.W. Heath. So Problem 10 is "negatively equivalent" to:

Problem 10′. *Is there a model of $2^{\aleph_0} < 2^{\aleph_1}$ in which there are no T_5 S-spaces?*

Indeed, every S-space has a subspace which is right-separable (hence scattered) in order type ω_1 and is also an S-space [10,3.1]. If the original space is T_5, then the subspace will have all the properties called for in Problem 10.

Here is one final problem which gets us back to compact spaces.

Problem 11. *Is it consistent that every separable compact T_5 space is of character $< c$?*

Under PFA, a counterexample would be generalized S-space, and I do not know of any compact generalized S-spaces under PFA, let alone T_5 ones.

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THE LINEARLY LINDELÖF PROBLEM

Mary Ellen Rudin

Does there exist a non-Lindelöf T_4 space X such that every increasing open cover of X has a countable subcover?

The question has remained unanswered for a least 30 years: see [1] and [2].

An open cover $\mathcal{U}$ is *increasing* if $\mathcal{U}$ can be indexed as $\{\mathcal{U}_\alpha \mid \alpha < \kappa\}$ for some ordinal κ with $\alpha < \beta < \kappa$ implying that $U_\beta \subset U_\alpha$.

An example X yielding a *yes* answer would have to be a Dowker space. If $\mathcal{V} = \{V_\alpha \mid \alpha < \kappa\}$ were an increasing open cover of X with $(V_\alpha - \cup_{\beta<\alpha} V_\beta) \neq 0$, then κ must have countable confinality. If A is a subset of X having regular uncountable cardinality, then A has a limit point x every neighborhood of which meets A in a set having the same cardinality.

No partial results are known.

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THE CARDINALITY OF LINDELÖF SPACES WITH POINTS G_δ

Franklin D. Tall

Arhangel'skiĭ raised the question of the cardinalitites of Lindelöf T_2 spaces with points G_δ and proved there are none of cardinality greater than or equal to the first measurable. Shelah proved there are none of weakly compact cardinality. Juhász [J$_1$] constructed such (non-T_2) spaces of arbitrarily large cardinality with countable confinality below the first measurable. Shelah showed it consistent with GCH that there is a 0-dimensional such space of size $\aleph_2$. He also showed it consistent from a weakly compact cardinal that $2^{\aleph_1} > \aleph_2$ and there is no such space of cardinality $\aleph_2$ (see [J$_2$]). Among other results in [T] we prove

Theorem 1. *Con (there is a supercompact) $\rightarrow$ Con($2^{\aleph_1}$ is arbitrarily large and there is no Lindelöf space with points G_δ of cardinality $\geq \aleph_2$ but $< 2^{\aleph_1}$).*

Theorem 2. *Con(there is a supercompact) $\rightarrow$ Con(GCH plus there is no indestructible Lindelöf space with points G_δ of cardinality $\geq \aleph_2$),*

where a Lindelöf space is *indestructible* if it cannot be destroyed by countably closed forcing.

The problem of finding a small consistent bound for the T_2 or for the first countable non-T_2 case remains open. It is not known whether such spaces can be destructible.

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BASIC PROBLEMS IN GENERAL TOPOLOGY

Stephen Watson

In 1936, Birkhoff published "On the Combination of Topologies" in Fundamenta Mathematicae [3]. In this paper , he ordered the family of all topologies on a set by letting $\tau_1 < \tau_2$ if and only if $\tau_1 \subset \tau_2$. He noted that the family of all topologies on a set is a lattice with a greatest element, the discrete topology and a smallest element, the indiscrete topology. The family of all T_1 topologies on a set is also a lattice whose smallest element is the cofinite topology whose proper closed sets are just the finite sets. Indeed, to study the lattice of all topologies on a set is to explore the fundamental interplay between general topology, set theory and finite combinatorics. Recent work has revealed some essential and difficult problems in the study of this lattice, especially in the study of complementation, a phenomena in these lattices akin in spirit to the study of Ramsey theory in combinatorial set theory. We say that topologies τ and σ are complementary if and only if $\tau \wedge \sigma = 0$ and $\tau \vee \sigma = 1$.

Problem 1 *Do there exist, in ZFC, more than $2^{\aleph_0}$ pairwise T_1-complementary topologies on the continuum?*

Anderson [1] showed in 1971 by a beautiful construction that there are at least κ pairwise complementary T_1 topologies on κ. In Steprāns and Watson [13], we showed that the maximum number of pairwise complementary T_1 topologies on ω is $\aleph_0$ and that it is consistent and independent whether there are $\aleph_2$ pairwise complementary T_1 topologies on ω_1. I believe that a solution will require a good mixture of finite combinatorics, topological intuition and hard set theory.

Problem 2. *Is there a linear lower bound for the maximum number of pairwise complementary partial orders on a finite set?*

These questions on complementation remain difficult in the context of finite T_0 topological spaces (also known as finite partial orders). This seems to be the most difficult and basic question on finite partial orders. To be exact, does there exist $\epsilon > 0$ such that , for any $n \in N$, there are at least $\epsilon \cdot n$ many pairwise complementary partial orders on a set of cardinality n? Specifically, we know [4] that there are at least $\frac{n}{80 log n}$ mutually complementary partial orders on n but cannot establish a linear lower bound.

Problem 3. *Can every lattice with 1 and 0 be homomorphically embedded as a sublattice in the lattice of topologies on some set?*

While these questions on complementation must be solved before some structural problems on these lattices can be attacked, there are some for which the tools may already exist. Problem 3 was solved affirmatively for finite lattices by Pudlak and Tuma in 1976 [?]. I conjecture that the answer is yes. Note that embedding the infinite lattice all of whose elements except 0 and 1 are incomparable means producing an infinite pairwise complementary family of topologies.

Problem 4. *Which lattics can be represented as the lattice of all topologies between two topologies? Can all finite lattices be represented in this fashion?*

The only non-trivial result I know on this problem is that the unique non-modular lattice of size 5 can be so represented.

Next, some fundamental questions on covering properties.

Problem 5. *Are Para-Lindelöf regular spaces countably paracompact?*

This is the main open problem on para-Lindelöf spaces. In 1981, Caryn Navy [9] was the first to construct a para-Lindelöf space which is not paracompact. Although her construction was quite flexible, it seems that it only produces spaces which are countably paracompact. Paradoxically, I believe that it is easiest to build a para-Lindelöf Dowker space! (see [15])

Another question which has not really been looked at but which I think is extremely important is:

Problem 6. *Are Para-Lindelöf collectionwise normal spaces paracompact?*

This was first asked by Fleissner and Reed in 1977 [7]. So far, there are no ideas at all on how to approach this. Even the much weaker property of meta-Lindelöf creates big problems here. The only consistent example of a meta-Lindelöf collectionwise normal space which is not paracompact seems to be Rudin's 1983 construction [12] under $V = L$ of a screenable normal space which is not paracompact.

Problem 7. *Does ZFC imply that there is a perfectly normal locally compact space which is not paracompact?*

This is my favorite question. If there is an example then it must be very strange. If there is such a space in ZFC, then, under $MA + \neg CH$, it is not collectionwise Hausdorff but, under $V = L$ it is collectionwise Hausdorff. There are examples in most models. Under CH, the Kunen line [8] is an example of a perfectly normal locally compact S-space which is not paracompact. Under $MA+\neg CH$, the Cantor tree with $\aleph_1$ branches is an example. On the other hand, a consistent theorem would be amazing. To reconcile the combinatiorics of $V = L$ with that of MA_{ω_1} would be hard and interesting.

Problem 8. *Are locally compact normal metacompact spaces paracompact?*

This was my thesis problem. It was asked by Arhangel'skiĭ in 1959 [2]. In 1980 [16], it as shown that, under $V = L$, locally compact normal metacompact spaces are paracompact. In 1983, Peg Daniels [5] raised everyone's hopes by showing in ZFC that locally compact normal *boundedly* metacompact spaces are paracompact by the problem remains open. If there is a theorem in ZFC it would be an astounding result. If there is an example in some model (as I think there is), it will require a deeper understanding of the Pixley-Roy space than so far exists.

A fundamental unsolved open question in the study of linearly ordered topological spaces is:

Problem 9. *Is there, in ZFC , a linear ordering in which every disjoint family of open sets is the union of countably many discrete subfamilies and yet in which there is no dense set which is the union of countably many closed discrete sets ? Is there such a linear ordering if and only if there is a Suslin line?*

A compact Suslin line is such a linear ordering but there may be others. The Urysohn metrization theorem is to the Nagata-Smirnov- Stone metrization theorem as the Suslin problem is to this problem. It is incredible that such a basic question about linear orderings is unsolved. In 1975, Nyikos [10] asked a closely related question: whether there is, in ZFC, a perfect non-archimedean space which is not metrizable. Purisch showed in 1983 [11] that perfect non-Archimedean spaces are orderable and Bennett and Lutzer [?] showed that an ordered space is perfect if and only if every disjoint family of open convex sets is the union of countably many discrete subfamilies.

Finally, a question on the combinatorial nature of connectedness.

Problem 10. *Is there a topological space (or a completely regular space) in which the connected sets (with more than one point) are precisely the cofinite sets?*

This question was motivated by an interesting paper of Tsvid [14]. In 1944, Erdös [6] attributed to Arthur Stone the result that there are no metrizable examples. In 1990, Gary Gruenhage used Martin's Axiom to construct a completely regular example but the ZFC question remains open. I conjecture that such an example exists in ZFC and that to establish this fact will depend on some hard finite combinatorics.

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