METRIZABLE GENERALIZED ORDER SPACES

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ABSTRACT. In 1971 D.J. Lutzer [10] proved a metrization theorem for generalized order topological spaces (GO-spaces) which says that, if $X$ is a $p$-embedded subspace of a linear ordered topological space, then $X$ is metrizable if and only if it has a $G_d$-diagonal. After stating this theorem, he raised the question whether there is any larger class of GO-spaces than the $p$-embedded subspaces of linear ordered topological spaces for which the $G_d$-diagonal metrization theorem is true. In this paper we answer this question negatively by proving the following result. If $(X, \leq, \tau)$ is a metrizable GO-space and $d$ is a metric on $X$ which is compatible with the topology $\tau$, then there is a metrizable linear ordered topological space $(Y, \leq_Y, \lambda)$ and a metric $d^*$ compatible with $\lambda$ such that (i) $(X, \leq)$ is a subordered set of $(Y, \leq_Y)$, (ii) $d^*$ is equivalent to $d$ on $X$ (equal if $d$ is bounded), and (iii) $(X, \tau)$ is a $p$-embedded closed subspace of $(Y, \lambda)$.

1. INTRODUCTION

Let $(X, \leq)$ be a linearly ordered set. We denote by

$$X(< a) = \{x \in X : x < a\} \quad \text{and} \quad X(> a) = \{x \in X : x > a\}$$

the open intervals determined by the element $a \in X$, and as usual $(a, b)$ denotes the open interval $X(> a) \cap X(< b)$. We also write $X(\leq a) = X(< a) \cup \{a\}$, and $X(\geq a)$ is similarly

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defined. The *linear order topology*, \( \lambda \), on \( X \) has for a subbasis the family of intervals

\[
B = \{X\} \cup \{X(< a) : a \in X\} \cup \{X(> a) : a \in X\}.
\]

A subspace of a linear ordered topological space (LOTS) is not, in general, a LOTS. For example, \( \mathbb{R} \), the real line with the natural ordering is a LOTS, but the subspace \( X = \{0\} \cup \{x : |x| > 1\} \) is not since \( \{0\} \) is an open set in the induced topology on \( X \), but not in the linear order topology on \( X \). A topology \( \tau \) on the linearly ordered set \((X, \leq)\) is called a *generalized order topology* on \( X \), briefly we say \((X, \leq, \tau)\) is a GO-space, if \( \tau \) extends the order topology and has a base of order-convex sets. An equivalent formulation, and the one we shall use, is that there are two subsets \( L, R \) of \( X \) such that, if \( a \in L \) then \( a \) is not the maximum element of \( X \) and \( X(\leq a) \) is open, and if \( a \in R \) then \( a \) is not the minimal element of \( X \) and \( X(\geq a) \) is open, and

\[
B \cup \{X(\leq a) : a \in L\} \cup \{X(\geq a) : a \in R\}
\]

is a subbasis for \( \tau \). A subspace of a LOTS is a GO-space.

A topology \( \tau \) on a set \( X \) is *metrizable* if there is a metric on \( X \) giving the same open sets. As an example, consider the metric space \((\mathbb{R}, d)\) on the real line illustrated in Diagram 1. In the diagram the segments \( A, B, C, D \) represent respectively the subintervals of the real line \((-\infty, 0], (0, 1], (1, 2], (2, \infty)\).
The metric is not the usual one for the real line, but the one induced by the distance in the plane. So, for example, the distance between the points $\varepsilon$ and $2 + \varepsilon$ is $2\varepsilon$ (if $0 < \varepsilon < 1$). Of course, this generalized order space (in which $L = \{0, 1, 2\}$ and $R = \emptyset$) is equivalent to that induced by the usual metric on $\mathbb{R}$ by the subspace $(-\infty, 0] \cup (1, 2] \cup (3, 4] \cup (5, \infty)$. In general, the structure of a GO-space is rather more complex.

During the last twenty years or so several papers have been written on the theory of LOTS and GO-spaces, and in particular about the metrization problem for such spaces. The first result in this direction was by V.V. Fedorčuk [7] who proved that a LOTS with a $\sigma$-locally countable base is metrizable. Then G. Creede [4] proved that a semi-stratifiable LOTS is metrizable. Shortly afterwards, D.J. Lutzer [9] generalized Creede's result by showing that a LOTS is metrizable if and only if it has a $G_\delta$-diagonal, in other words if $\Delta = \{(x, x) : x \in X\}$ is a $G_\delta$-set in the product space $X \times X$; of course, any metric space has a $G_\delta$-diagonal. Also, M.J. Faber [5] used some classical theorems of R.H. Bing to obtain metrization theorems for LOTS.

D.J. Lutzer [10] was the first to consider subspaces of LOTS, i.e. GO-spaces, and he established the following sufficient condition for a subspace of a LOTS to be metrizable.

**Theorem 1.1.** Let $(Y, \leq, \lambda)$ be a LOTS and let $\tau$ be the relative topology on a p-embedded subspace $X$. If $(X, \tau)$ has a $G_\delta$-diagonal, then $(X, \tau)$ is metrizable.

Recall that the space $X$ is a $p$-embedded subspace of $Y$ if there is a sequence $(\mathcal{U}(n) : n < \omega)$ of covers of $X$ by open subsets of $Y$ such that, for each $x \in X$, 

$$\bigcap_{n<\omega} \text{St}(x, \mathcal{U}(n)) \subseteq X,$$

where $\text{St}(x, \mathcal{U}(n)) = \bigcup\{U \in \mathcal{U}(n) : x \in U\}$.

M.J. Faber [5], [6], J.M. van Wouwe [12], [13], and H. Bennett & D.J. Lutzer [1] obtained various necessary and sufficient conditions for a GO-space to be metrizable, and H. Bennett in
used some of these results to give another proof of an observation of S. Purisch ([11] Propositions 2.4 and 2.5) that there is a metric $\rho$ on the GO-space $(X, \leq, \tau)$ which is compatible with the topology $\tau$ and respects the order in the sense that

$$x \leq y \leq z \Rightarrow \rho(x, y) \leq \rho(x, z).$$

(Note that the metric on $\mathbb{R}$ described in diagram 1 does not respect the order.) More recently, H. Bennett [3] improved Lutzer's theorem by proving that a LOTS with an $S_\delta$-diagonal is metrizable.

In this paper we settle a question raised by D.J. Lutzer in [10]. After the statement of Theorem 1.1 in [10], Lutzer remarked that he did not know of any class of GO-spaces larger than the $p$-embedded subspaces of LOTS for which the $G_\delta$-metrization theorem is true. We show that there is no larger class. In other words, if $(X, \leq, \tau)$ is a metrizable GO-space, there is some LOTS $Y$ such that $X$ is a $p$-embedded induced subspace. In fact, there is a metrizable LOTS $Y$. We prove the following theorem.

**Theorem 1.2.** If $(X, \leq, \tau)$ is a metrizable generalized order space with metric $d$, then there is a metrizable LOTS $(Y, \leq, \lambda)$ with metric $d^*$ such that (i) $\leq_X = \leq_Y |X \times X$, (ii) $d^*$ is equivalent to $d$ on $X$ (equal to $d$ on $X$ if $d$ is bounded), and (iii) $X$ is a $p$-embedded closed subspace of $Y$.

As a corollary of Theorems 1.1 and 1.2 we have a necessary and sufficient condition for a GO-space to be metrizable.

**Theorem 1.3.** A GO-space is metrizable if and only if it is a $p$-embedded closed subspace of a metrizable LOTS.

## 2. ARC-CONNECTED EXTENSION OF A METRIC SPACE

In order to prove Theorem 1.2 we need a result about arc-connected metric spaces. A topological space $(X, \tau)$ is arc-connected if for any two distinct points $a, b \in X$ there is a homeomorphic map $f : [0,1] \to X$ such that $f(0) = a$.
and \( f(1) = b \). The following theorem shows that a metric space can be isometrically embedded in an arc-connected metric space. In fact, for our application we shall require the result for pseudo-metric spaces, i.e. when the metric \( d : X \times X \to \mathbb{R} \) is non-negative, symmetric and satisfies the triangle inequality, but we do not insist that \( d(x, y) = 0 \Rightarrow x = y \). Of course, if \((X, d)\) is a pseudo-metric space and we define an equivalence relation \( \sim \) on \( X \) by \( x \sim y \iff d(x, y) = 0 \), then \( X/\sim \) is a metric space with the induced metric. Theorem 2.1 is proven in ([8, page 81]) for bounded metric spaces (which is the essential content). We give the details of the proof since we require the result for pseudo-metrics and we continue to use the notation introduced in the proof.

**Theorem 2.1.** If \((X, d)\) is a (pseudo-)metric space, then there is an arc-connected (pseudo-)metric space \((X^*, d^*)\) such that \((X, d)\) can be isometrically embedded into \((X^*, d^*)\).

**Proof:** Let \( < \) be a linear ordering of \( X \). For distinct elements \( a, a' \in X \) with \( a < a' \) we introduce a copy of the open unit interval \( I(a, a') = \{ x \in (a, a') : 0 < \lambda < 1 \} \); we also define \( x_0(a, a') = a \) and \( x_1(a, a') = a' \). We assume that \( I(a, a') \cap I(b, b') = \emptyset \) if \( (a, a') \neq (b, b') \), and define \( X^* = X \cup \{ I(a, b) : a, a' \in X, a < a' \} \). We define a (pseudo-)metric \( d^* \) on \( X^* \) by setting, for \( x = x_\lambda(a, a') \) and \( y = x_\mu(b, b') \),

\[
d^*(x, y) = \begin{cases} |\lambda - \mu|d(a, a') & \text{if } (b, b') = (a, a') \\ \lambda'\mu'd(a, b) + \lambda\mu'd(a, b') + \lambda\mu'd(a', b') & \text{if } (b, b') \neq (a, a') \end{cases}
\]

where we have written \( \lambda' = 1 - \lambda \), \( \mu' = 1 - \mu \).

It is easy to check that \( d^* \) is unambiguously defined. For example, using the second line of the definition to compute the distance \( d^*(x_\lambda(a, a'), a) = d^*(x_\lambda(a, a'), x_1(c, a)) \), where \( c \neq a \), we get (since \( \mu = 1, \mu' = 0 \))

\[
\lambda'd(a, a) + \lambda d(a', a) = \lambda d(a, a'),
\]
and this is the same as the value that we obtain using the first line.

Note that, if \( b \in X \), then
\[
d^*(x_\lambda(a, a'), b) = \lambda'd(a, b) + \lambda d(a', b).
\]
Also, if \((a, a') \neq (b, b')\) then
\[
d^*(x_\lambda(a, a'), x_\mu(b, b')) = \lambda'd^*(a, x_\mu(b, b')) + \lambda d^*(a', x_\mu(b, b'))
\]
\[
(2) = \mu d^*(x_\lambda(a, a'), b') + \mu' d^*(x_\lambda(a, a'), b).
\]

To show that \(d^*\) is a (pseudo-) metric is a little tedious. It is obvious that \(d^*\) is symmetric. Also, if \(d\) is a metric, then \(d^*(x, y) = 0 \iff x = y\). We have to check that the triangle inequality holds.

**Case 1:** If \(x = x_\lambda(a, a'), y = x_\mu(a, a'), z = x_\nu(a, a')\), it is obvious that \(d^*(x, z) \leq d^*(x, y) + d^*(y, z)\).

**Case 2:** Let \(x = x_\lambda(a, a'), y = x_\mu(b, b'), z = x_\nu(c, c')\), where \((a, a'), (b, b')\) and \((c, c')\) are all different. We have
\[
d^*(x, y) = (\lambda'\mu'\nu d(a, b) + \lambda'\mu d(a, b')) + \lambda d(a', b'))(\nu + \nu')
\]
\[
\leq \lambda'\mu'\nu'(d(a, c) + d(b, c)) + \lambda'\mu'\nu'(d(a, c') + d(b, c'))
\]
\[
+ \lambda'\mu'\nu'(d(a', c) + d(b', c)) + \lambda'\mu'\nu'(d(a', c') + d(b', c'))
\]
\[
+ \lambda\mu\nu'(d(a', c) + d(b', c)) + \lambda\mu\nu'(d(a', c') + d(b', c'))
\]
\[
= (\lambda'\nu d(a, c) + \lambda'\nu' d(a', c) + \lambda'\nu d(a, c') + \lambda'\nu' d(a', c'))
\]
\[
+ \mu'\nu d(b, c) + \mu'\nu' d(b', c) + \mu'\nu d(b, c') + \mu'\nu' d(b', c')
\]
\[
= d^*(x, z) + d^*(y, z).
\]

**Case 3:** Let \(x = x_\lambda(a, a'), y = x_\mu(a, a'), z = x_\nu(c, c')\), where \((a, a')\) and \((c, c')\) are different. We need to verify that the following two inequalities hold:

\[
(3) d^*(x, y) \leq d^*(x, z) + d^*(y, z)
\]
\[
(4) d^*(y, z) \leq d^*(x, y) + d^*(x, z).
\]
First we show that (3) and (4) hold in the special case when 
\( \lambda = 0, \mu = 1 \), i.e. when \( x = a, y = a' \). For these special values we have

\[
\begin{align*}
d(a, a') & \leq \nu(d(a, c') + d(a', c')) + \nu'(d(a, c) + d(a', c)) \\
& = d^*(a, z) + d^*(a', z),
\end{align*}
\]

and

\[
\begin{align*}
d^*(a', z) & = \nu'd(a', c) + \nu d(a', c') \\
& \leq \nu'(d(a, a') + d(a, c)) + \nu(d(a, a') + d(a, c)) \\
& = d(a, a') + d^*(a, z).
\end{align*}
\]

(3) and (4) follow from these special cases. For (3) we may assume that \( \lambda < \mu \). Then, since \( \mu - \lambda \leq \min\{\mu + \lambda, \mu' + \lambda'\} \), it follows that

\[
\begin{align*}
d^*(x, y) & = (\mu - \lambda)d(a, a') \leq (\mu - \lambda)(d^*(a, z) + d^*(a', z)) \\
& \leq (\mu' + \lambda')d^*(a, z) + (\mu + \lambda)d^*(a', z) \\
& = (\mu' + \lambda')(\nu'd(a, c) + \nu d(a, c')) + (\mu + \lambda)(\nu'd(a', c) + \nu d(a', c')) \\
& \quad + \nu d(a', c') \\
& = (\lambda'\nu'd(a, c) + \lambda\nu d(a', c) + \lambda'\nu d(a, c') + \lambda\nu d(a', c')) \\
& \quad + (\mu'\nu'd(a, c) + \mu\nu d(a', c) + \mu'\nu d(a', c') + \mu\nu d(a', c')) \\
& = d^*(x, z) + d^*(y, z).
\end{align*}
\]

This proves (3). We prove (4) under the same assumption that \( \lambda < \mu \) (the case when \( \mu < \lambda \) is similar). By (2), we have

\[
\begin{align*}
d^*(y, z) & = \mu'd^*(a, z) + \mu d^*(a', z) \\
& = (\mu - \lambda)d^*(a', z) + \lambda d^*(a', z) + \mu'd^*(a, z) \\
& \leq (\mu - \lambda)(d^*(a, a') + d^*(a, z)) + \lambda d^*(a', z) + \mu'd^*(a, z) \\
& = (\mu - \lambda)d^*(a, a') + \lambda'd^*(a, z) + \lambda d^*(a', z) \\
& = d^*(x, y) + d^*(x, z).
\end{align*}
\]

Clearly the space \((X^*, d^*)\) is an arc-connected isometric extension of \((X, d)\). For example, if \( x = x_\lambda(a, a') \), \( y = x_\mu(b, b') \), where \((a, a') \neq (b, b')\) and \( a < b \), then there is a homeomorphic
map $f : [0, 1] \rightarrow \{x_\nu(a, a') : \nu \leq \lambda\} \cup I(a, b) \cup \{x_\mu(b, b') : \mu \leq \mu\}$ with $f(0) = x$, $f(1) = y$. □

We call the (pseudo-)metric space $(X^*, d^*)$ constructed in the theorem the arc-connected extension of $(X, d)$. It should be noted that the linear ordering imposed upon $X$ in the proof was no more than a notational convenience, the construction of $(X^*, d^*)$ does not depend upon this ordering. In the case when $(X, d)$ is a pseudo-metric space, then so also is $(X^*, d^*)$. But in this case it is clear from our definitions that, if $a, a', b, b' \in X$, $a \neq a'$, $b \neq b'$, then:

1. If $d(a, a') = 0$ and $x, y \in I(a, a')$, then $d^*(x, y) = 0$.
2. If $d(a, a') = d(a, b) = d(a', b') = 0$, $x \in I(a, a')$, $y \in I(b, b')$, then $d^*(x, y) = 0$.
3. If $d(a, a') = 0$, $x \in I(a, a')$, $y \in I(b, b')$ and $d^*(a, y) > 0$, then $d^*(x, y) > 0$.

Corollary 2.2. If $(X^*, d^*)$ is the arc-connected extension of the pseudo-metric space $(X, d)$, and if $d(a, a') \neq 0$ and $x \in I(a, a')$, then there is $r > 0$ such that $B^*(x, r) = \{y \in X^* : d^*(x, y) < r\} \subseteq I(a, a')$.

Proof: Let $x = x_\lambda(a, a')$, where $0 < \lambda < 1$. Choose $r$ so that $0 < r < r' < \min\{\lambda d(a, a'), \lambda' d(a, a')\}$. Then $d^*(a, x) > r$, $d^*(a', x) > r$. Also, if $y = x_\mu(b, b')$, where $(b, b') \neq (a, a')$, then

$$d^*(x, y) = \lambda' \mu'd(a, b) + \lambda \mu'd(a', b') + \lambda \mu'd(a', b) + \lambda \mu d(a', b')$$
$$\geq (\mu'(d(a, b) + d(a', b')) + \mu(d(a, b'))$$
$$+d(a', b'))r'/d(a, a') \geq r' > r,$$

and the result follows. □

From Corollary 2.2 we immediately obtain the following fact.

Corollary 2.3. Let $(X, d)$ be a pseudo-metric space with arc-connected extension $(X^*, d^*)$. Let $X' \subseteq X$ be a set such that $\{a, a'\} \cap X' \neq \emptyset$ whenever $a \neq a'$ and $d(a, a') = 0$, and let $\hat{X} = \bigcup\{I(a, a') : a \neq a' \in X, d(a, a') = 0\}$. Then $d^*(x, y) > 0$
for \( x \neq y \) and \( x, y \in X^{**} = X^* (X' \cup \hat{X}) \), i.e the subspace \( X^{**} \) is a metric space.

We conclude this section with the observation that the arc-connected extension of a metric space reflects completeness.

**Theorem 2.4.** A metric space is complete if and only if its arc-connected extension is complete.

**Proof:** Let \((X^*, d^*)\) be the arc-connected extension of the metric space \((X, d)\). Suppose \(X^*\) is complete. Then, if \((a_n)\) is a Cauchy sequence in \(X\), there is \(x \in X^*\) such that \(a_n\) converges to \(x\). By Corollary 2.2 it follows that \(x \in X\), and so \(X\) is complete.

Now suppose that \(X\) is complete. Let \((y_n)\) be a Cauchy sequence in \(X^*\), \(y_n = x_{\lambda_n}(a_n, b_n)\). We need to show that some subsequence of \((y_n)\) converges. Suppose \(\liminf \lambda_n = 0\); we can assume that \(\lambda_n \to 0\). Since \(d^*(a_n, y_n) \to 0\) it follows from the triangle inequality that \((a_n)\) is also Cauchy and so converges to some \(a \in X\). Since \(d^*(a_n, a)\) and \(d^*(a_n, y_n)\) both converge to 0, it follows that \(y_n \to a\). A similar argument applies if \(\limsup \lambda_n = 1\). Thus we may assume that (some subsequence) \(\lambda_n \to p\) where \(0 < p < 1\). By Corollary 2.2 it follows that the pairs \((a_n, b_n)\) are eventually constant, say equal to \((a, b)\). Then \(y_n \to x_p(a, b)\). \(\square\)

3. **Proof of Theorem 1.2**

Let \((X, \leq, \tau)\) be a metrizable GO-space. We may assume that the metric \(d\) on \(X\) which is compatible with \(\tau\) is bounded. Let \(L = \{x \in X : X(\leq x) \text{ is open} \} \setminus \{\max X\}, R = \{x \in X : X(\geq x) \text{ is open} \} \setminus \{\min X\}\).

If the element \(x \in X\) has an immediate successor in the ordering on \(X\), we denote its successor by \(x^+\); similarly if there is an immediate predecessor we denote it by \(x^-\). If \(x \in L\) has no immediate successor in \((X, \leq)\), then we extend the order by introducing a new element \(x^+\) which is the immediate successor of \(x\) in the extended order. Similarly, for each element of \(R\)
which has no immediate predecessor in the order on $X$, we introduce one which we denote by $x^-$. Let $(X', \leq)$ be the extended ordered set which includes these additional elements $x^+$ or $x^-$ for appropriate elements $x \in L \cup R$. Thus each element of $L$ has an immediate successor and each element of $R$ has an immediate predecessor in this extended order.

We define a symmetric non-negative real function $d' : X' \times X' \to \mathbb{R}$ as follows: for $x, y \in X'$,

$$d'(x, y) = \begin{cases} 
  d(x, y) & \text{if } x, y \in X; \\
  \inf v \in X(> a) a < u < v \sup d(x, u) & \text{if } x \in X, a \in L, \\
  \inf v \in X(< a) v < u < a \sup d(x, u) & \text{if } y = a^+ \notin X; \\
  \inf v \in X(> a) a < u < v \sup d(u, t) & \text{if } y = a^- \notin X; \\
  \inf w \in X(< b) b < t < w \sup d(x, u) & \text{if } a, b \in L, \\
  \inf w \in X(< b) b < t < w \sup d(u, t) & \text{if } x = a^+ \notin X, \\
  \inf w \in X(< b) b < t < w \sup d(x, u) & \text{if } y = b^+ \notin X.
\end{cases}$$

There are similar definitions for $d'(a^+, b^-)$ and $d'(a^-, b^-)$ obtained by modifying the last line of the above in an obvious manner.

We first observe that

(5) \hspace{1cm} d'(x, y) > 0 \text{ if } x \in X, y \in X' \setminus X.

We only prove this for the case when $y = a^+$ for some $a \in L$ which has no immediate successor in $X$; the case when $y = a^-$ for some $a \in R$ is similar. Suppose $x < a$. Since $X(\leq a)$ is open and the metric $d$ is compatible with $\tau$, there is $r > 0$ such that $B_X(x, r) = \{ y \in X : d(x, y) < r \} \subseteq X(\leq a)$. Thus $d(x, u) \geq r$ for all $u \in X(> a)$ and since $X(> a) \neq \emptyset$ it follows that $d'(x, y) \geq r$. Now suppose that $x > a$. Since $X(> a)$ has no first element in the ordering of $X$, there is some $v \in X$ such that $a < v < x$. Then $X(> v)$ is an open neighbourhood of $x$ and so there is some $r > 0$ such that $B_X(x, r) \subseteq X(> v)$. This implies that $d'(x, y) \geq r$ and (5) follows.
We now verify that $d'$ is a pseudo-metric on $X'$. Since $d'$ is symmetric by definition, we need only check that the triangle inequality

\[ d'(x, z) \leq d'(x, y) + d'(y, z), \tag{6} \]

holds for distinct $x, y, z \in X'$. There are several different cases that need to be considered, but these are all rather similar, and to avoid trivial repetition when we consider a point, say $x$, in $X' \setminus X$ we assume $x = a^+$ for some $a \in L$.

**Case 1.** $x, z \in X$, $y \in X' \setminus X$.

Assume $y = a^+$ for some $a \in L$. Then, for $a < u < v$, $u, v \in X$, we have

\[
d'(x, z) = d(x, z) \leq d(x, u) + d(u, z)
\leq \sup\{d(x, u) : a < u < v\} + \sup\{d(u, z) : a < u < v\}
\]

and hence (6) holds.

**Case 2.** $x, y \in X$, $z \in X' \setminus X$.

Assume $z = a^+$ for some $a \in L$. For $a < u < v$, $u, v \in X$ we have

\[
d(x, u) \leq d(x, y) + d(y, u)
\]

and so, taking the supremum of both sides for $u < v$, we have

\[
\sup\{d(x, u) : a < u < v\} \leq d(x, y) + \sup\{d(y, u) : a < u < v\}.
\]

Finally taking the infimum of both sides of this for $v > a$ we get (6).

**Case 3.** $x \in X$; $y, z \in X' \setminus X$.

Assume $y = a^+$, $z = b^+$ for some $a, b \in L$. For $a < u < v$, $b < u' < v'$ and $u, v, u', v' \in X$ we have

\[
\begin{align*}
d(x, u') & \leq d(x, u) + d(u, u') \\
& \leq \sup\{d(x, u) : a < u < v\} \\
& \quad + \sup\{d(u, u') : a < u < v\},
\end{align*}
\]

and hence

\[
\begin{align*}
\sup\{d(x, u') : b < u' < v'\} & \leq \sup\{d(x, u) : a < u < v\} \\
& \quad + \sup\{d(u, u') : a < u < v, b < u' < v'\}.
\end{align*}
\]
Taking the infimum over \( v > a \) and \( v' > b \), (6) follows.

**Case 4.** \( x, z \in X' \setminus X, \ y \in X \). This is similar to Case 3.

**Case 5.** \( x, y, z \in X' \setminus X \).

Assume \( x = a^+, \ y = b^+, \ z = c^+ \) for some \( a, b, c \in L \). Let \( a < u < v, \ b < u' < v', \ c < u'' < v'' \). We have

\[
d(u, u'') \leq d(u, u') + d(u', u'')
\]

\[
\leq \sup \{d(u, u') : a < u < v, b < u' < v'\}
+ \sup \{d(u', u'') : b < u' < v', c < u'' < v''\},
\]

and therefore,

\[
\sup \{d(u, u'') : a < u < v, c < u'' < v''\}
\leq \sup \{d(u, u') : a < u < v, b < u' < v'\}
+ \sup \{d(u', u'') : b < u' < v', c < u'' < v''\}.
\]

Taking the infimums of the terms on the left and right sides of this inequality gives (6).

This proves that \( d' \) is a pseudo-metric on \( X' \). Unfortunately, it need not be a metric. To see this consider again the example illustrated in Diagram 1. In that example, \( L = \{0, 1, 2\} \), \( R = \emptyset \), and we have to adjoin the additional points \( 0^+, 1^+ \) and \( 2^+ \). The distance between the distinct points \( 0^+ \) and \( 2^+ \) is

\[
d'(0^+, 2^+) = \inf_{0 < \xi \leq 2} \sup \{d(\xi, 2 + \eta) : 0 < \xi < \epsilon, 0 < \eta < \epsilon\} = 0.
\]

However, by (5), the set \( Z = \{x \in X' : (\exists y \neq x) d'(x, y) = 0\} \subseteq X' \setminus X \).

By Theorem 2.1 there is an arc-connected extension \((X^*, d^*)\) of the pseudo-metric space \((X', d')\). Also, by Corollary 2.3 the subspace \( X^{**} \) is a metric space, where \( X^{**} = X^* \setminus \hat{X} \) and \( \hat{X} = \cup \{I(a, a') \cup \{a, a'\} : a \neq a' \in X', d'(a, a') = 0\} \). Here we use the same notation as in the proof of Theorem 2.1 so that \( I(a, a') = \{x_\lambda(a, a') : 0 < \lambda < 1\} \) for points \( a, a' \in X' \) with \( a < a' \).

We now show that (5) extends to the following:

\[
d^*(x, y) > 0 \text{ if } x = x_\lambda(a, a'), a \in X, 0 \leq \lambda < 1 \text{ and } y \in X^* \setminus \{x\}.
\]
For, let \( y = x_\mu(b,b') \), where \( b, b' \in X' \) and \( 0 \leq \mu \leq 1 \). If \((b,b') = (a,a')\), then \( \mu \neq \lambda \) and \( d^*(x,y) = |\lambda - \mu|d'(a,a') > 0 \) by (5). Also, if \((b,b') \neq (a,a')\), then \( d^*(x,y) \geq \lambda'(\mu'd(a,b) + \mu d(a,b')) > 0 \) again by (5).

It follows from (7) that \( Y = X \cup L^* \cup R^* \) is disjoint from \( X' \), where \( L^* = \bigcup \{ I(x,x^+) : x \in L \} \), \( R^* = \bigcup \{ I(x^-,x) : x \in R \} \).

Hence the restriction of \( X^{**} \) to \( Y \) is also a metric space.

We define a linear ordering \( \leq_Y \) of \( Y \) as follows:

\[
x \leq_Y y \iff \begin{cases} 
  x \leq y & \text{when } x, y \in X \\
  x \leq a & \text{when } x \in X, a \in L \text{ and } y \in I(a,a^+) \\
  x \leq a & \text{when } x \in X, a \in R \text{ and } y \in I(a^-,a) \\
  a \leq y & \text{when } y \in X, a \in L \text{ and } x \in I(a,a^+) \\
  a \leq y & \text{when } y \in X, a \in R \text{ and } x \in I(a^-,a) \\
  a \leq b & \text{when } a, b \in L \cup R, x \in I(a,a^+) \text{ or } I(a^-,a), \text{ and } y \in I(b,b^+) \text{ or } I(b^-,b) \\
  \lambda < \mu & \text{when } a \in L, x = x_\lambda(a,a^+) \\
  y = x_\mu(a,a^+) \text{ or } a \in R, x = x_\lambda(a^-,a), \\
  y = x_\mu(a^-,a)
\end{cases}
\]

It is easy to check that \( \leq_Y \) is a linear order which extends the order on \( X \), and also that, for \( a \in L \) and \( b \in R \), \( I(a,a^+) \) and \( I(b^-,b) \) are intervals in \((Y, \leq_Y)\). (As observed by the referee, the order on \( Y \) is more easily visualized if we identify \( Y \) with \( [X \times \{0\}] \cup [L \times (0,1)] \cup [R \times (-1,0)] \), and then \( \leq_Y \) is just the order inherited from the lexicographic order on \( X \times (-1,1) \).)

To complete the proof of the theorem we need to show two things: (A) the metric \( d^* \) is compatible with the linear order topology on \( Y \); (B) \( X \) is a \( p \)-embedded, closed subspace of \( Y \).

**Proof of (A):** We first show that the linear order topology on \( Y \) is contained in the metric topology defined by the metric \( d^* \). Let \( z \in Y \) and let \( J \) be an open interval in the order topology on \( Y \) which contains \( z \). We have to show that there is \( r > 0 \) such that the open ball \( B_Y(z,r) = \{ y \in Y : d^*_Y(y,z) < r \} \) is contained in \( J \).

If \( z \in L^* \), then there is \( a \in X \) such that \( z \in I(a,a^+) \). By
Corollary 2.2 there is \( r' > 0 \) such that \( B_Y(z, r') \subseteq I(a, a^+) \) and hence there is \( r > 0 \) such that \( B_Y(z, r) \subseteq J \). Similarly if \( z \in R^* \). Thus we may assume that \( z \in X \). We need to consider several different cases.

Suppose that \( z \in L \setminus R \). Since \( J \) is an open interval of \( Y \), we may assume that \( J \cap Y(> z) \subseteq I(z, z^+) \) so that \( J \cap X \subseteq X(\leq z) \) is an open neighbourhood of \( z \) in \( X \). Since \( z \notin R \), \( \{z\} \) is not open in \( X \) and so there is some element \( b \in X(< z) \) such that \((b, z) \cap X \subseteq J \). Thus we may assume that \( J = (b, c) \), where \( b \in X \) and \( b < z < c \in I(z, z^+) \). Since the metric \( d \) is compatible with the topology \( T \) on \( X \), there is \( r_1 > 0 \) such that \( B_X(z, r_1) \subseteq J \cap X \). Also, there is \( r_2 > 0 \) such that \( B_Y(z, r_2) \cap Y(\geq z) \subseteq J \). We claim that \( B_Y(z, r) \cap X = B_X(z, r) \) we need only show that \( B_Y(z, r) \cap (X \cup I(z, z^+)) \subseteq J \).

Let \( y \in B_Y(z, r) \cap (X \cup I(z, z^+)) \). We consider only the case when \( y = x_\lambda(a, a^+) \) for some \( a \in L \) and \( 0 < \lambda < 1 \); the other case when \( y = x_\lambda(a^-, a) \) for some \( a \in R \) is similar. Clearly \( a < z \) since \( B_Y(z, r) \cap Y(\geq z) \subseteq I(z, z^+) \). Suppose that \( a < b \). It follows from the definition of \( d^* \) (see Proof of Theorem 2.1) that \( d^*(y, z) = \lambda d'(a, z) + \lambda d'(a^+, z) \geq \min\{d'(a, z), d'(a^+, z)\} \). Now \( d'(a, z) = d(a, z) \geq r \) since \( B_X(z, r) \subseteq \{x \in X : b < x \leq z\} \). Also,

\[
    d'(z, a^+) = \inf_{u \in X(a)} \sup_{v \in X(>a)} \{d(z, u) : a < u < v\} \geq r.
\]

This is true since if \( b < v \), then \( \sup\{d(z, u) : a < u < v\} \geq d(z, b) \geq r \), and if \( a < v \leq b \), \( d(z, u) \geq r \) for all \( u \in X \) such that \( a < u < v \). Thus \( d^*(y, z) \geq r \). This is a contradiction and hence \( b \leq a \). It follows that \( y \in J \) since \( b \leq a <_Y y \leq_Y z \) and \( b, z \) are elements in the interval \( J \) of \( Y \).

The case \( z \in R \setminus L \) is similar. The case \( z \in L \setminus R \) is simpler since, in this case, \( \{z\} \) is open in \( X \), and we may assume that \( J \subset (z^-, z^+) \) and so there is \( r > 0 \) such that \( B_Y(z, r) \subseteq J \).

Finally, suppose that \( z \in X \setminus (L \cup R) \). Since neither \( X(\geq z) \) nor \( X(\leq z) \) is open, it follows that there are \( b, c \in J \cap X \) such that \( b < z < c \). Thus we may assume that \( J = (b, c) \). Since
(b, c) \cap X is an open neighbourhood of z in X, there is r > 0 such that $B_X(z, r) \subseteq J$. Then by a similar argument to the one above it follows that $B_Y(z, r) \subseteq J$.

We now prove the converse, that the metric topology on $Y$ is contained in the linear order topology on $Y$. We have to show that, for any $z \in Y$ and $r > 0$, there are $b, c \in B_Y(z, r)$ such that $b <_Y z <_Y c$ and $x \in B_Y(z, r)$ whenever $b <_Y x <_Y c$.

If $z \in Y \setminus X$, say $z \in I(a, a^+)$ for some $a \in L$, the result is obvious since, by Corollary 2.2 there is $r_1$ such that $0 < r_1 < r$ and $B_Y(z, r_1) \subseteq I(a, a^+)$ and $B_Y(z, r_1)$ is an interval in $(Y, \leq_Y)$.

Suppose $z \in X$. We only consider the case when $z \in X \setminus (L \cup R)$; the other cases are similar. Since $x \notin L \cup R$, and since the metric $d$ on $X$ is compatible with the generalized order topology on $X$, it follows that there are $r > 0$ and $b, c \in X$ such that $b < z < c$ and $\{y \in X : b \leq y \leq c\} \subseteq X \cap B_Y(z, r/2)$.

We will show that $d^*(y, z) < r$ holds for all $y \in Y$ such that $b <_Y y <_Y c$. If $y \in X$ this is clear. Suppose $y \in Y \setminus X$, say $y = x_\lambda(a, a^+)$ for some $a \in L$ and $0 < \lambda < 1$. If $a < b$ then we get the contradiction that $y <_Y b$. Therefore, $b \leq a$. Similarly, $a < c$. Hence $d(a, z) \leq r/2$. If $a^+ \in X$, then $b < a^+ \leq c$ and so $d(a^+, z) \leq r/2$; on the other hand, if $a^+ \notin X$ then $d'(a^+, z) \leq \sup\{d(z, u) : u \in X, a < u < z\} \leq r/2$. In any case, $d^*(y, z) = \lambda d(a, z) + \lambda d'(a^+, z) < r$. This completes the proof of (A).

Proof of (B): Clearly $(X, \tau)$ is a subspace of $Y$ and it is closed since the sets $I(a, a^+) (a \in L)$ and $I(a^-, a) (a \in R)$ are open intervals of $Y$.

For any positive integer $n$ let $U(n) = \{B_Y(x, \frac{1}{2n}) : x \in X\}$. Then $U(n)$ is a cover of $X$ by open subsets of $Y$. Also, for $x \in X$, we have $St(x, U(n)) \subseteq B_Y(x, \frac{1}{n})$, and so $\bigcap_{n \geq 1} St(x, U(n)) \subseteq \bigcap B_Y(x, \frac{1}{n}) = \{x\} \subseteq X$. Thus $X$ is a $p$-embedded closed subset of $Y$. This completes the proof of the theorem. \qed
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